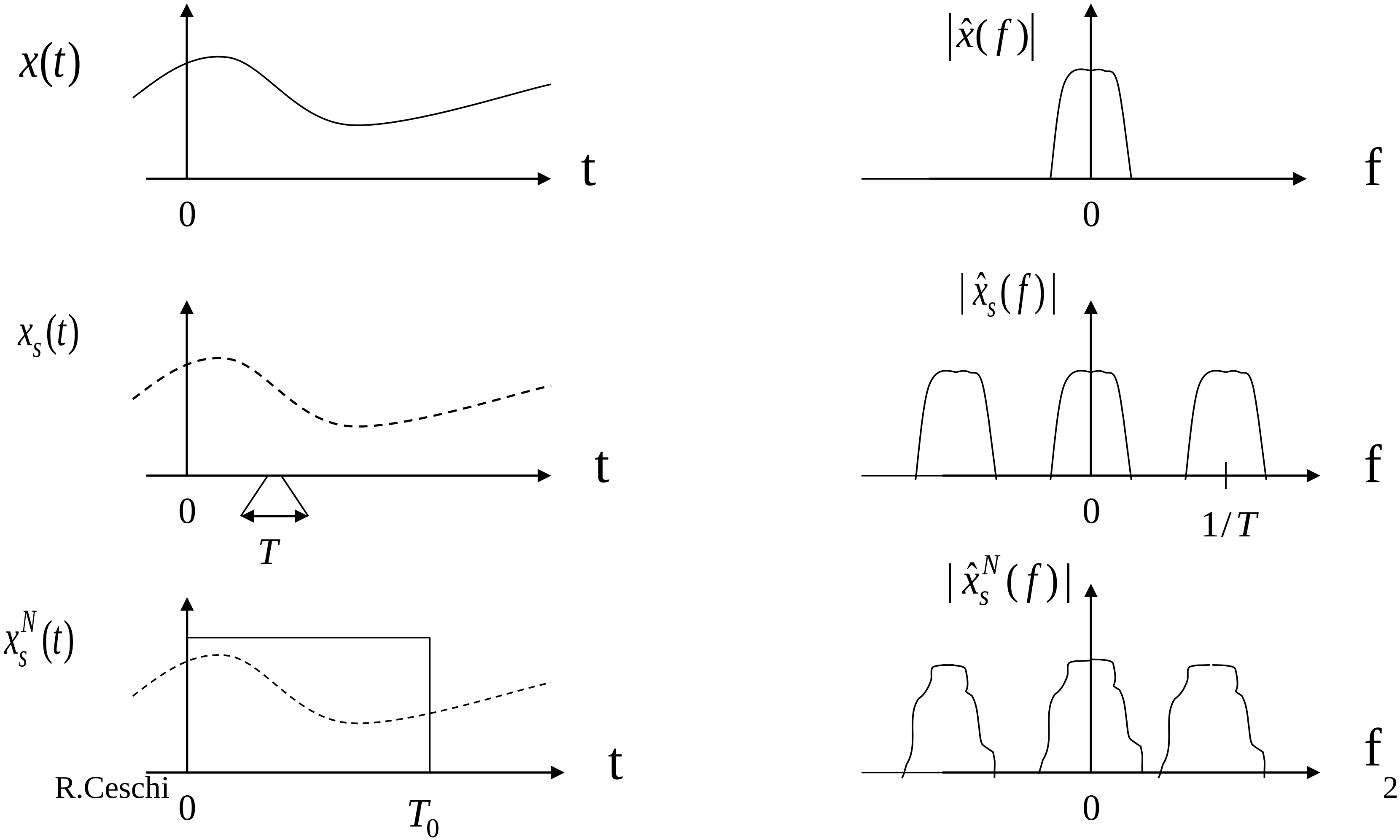
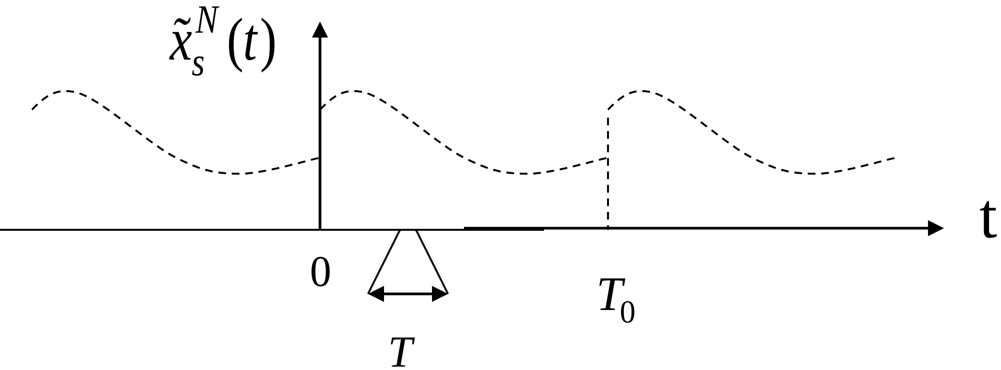
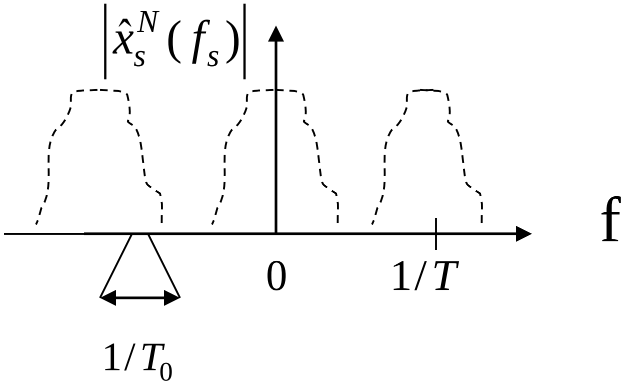
Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT)

Discrete Fourier transform



DFT (suite)





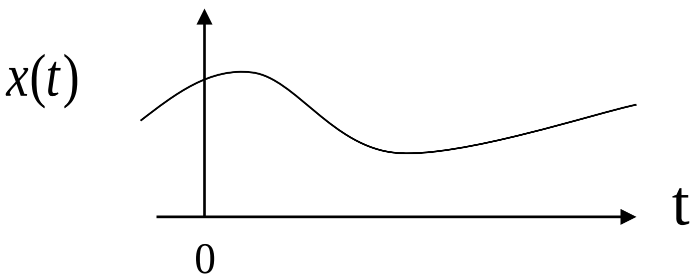
$$\tilde{x}_s^N(t) = \tilde{x}_s^N(t+kT_0) \Rightarrow$$

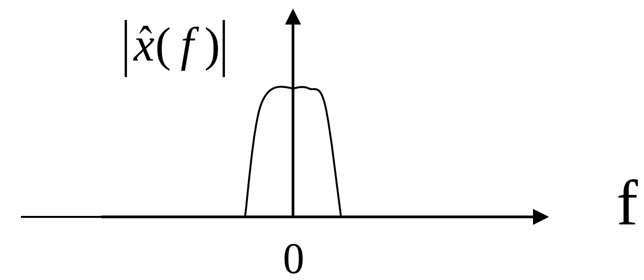
$$\tilde{\hat{x}}_s^N(f_s) = \tilde{\hat{x}}_s^N(f_s = \frac{k}{T_0}) \Rightarrow$$

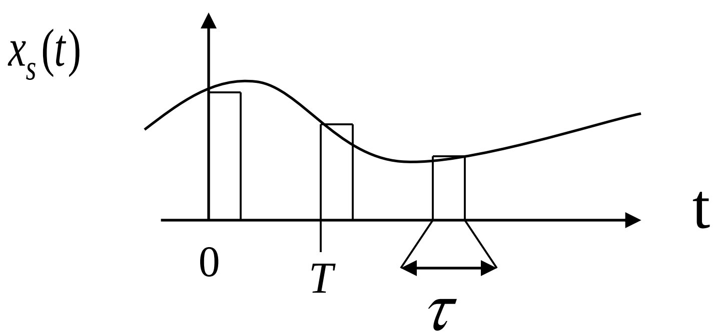
$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}_k . e^{\frac{2j\pi kn}{N}}$$

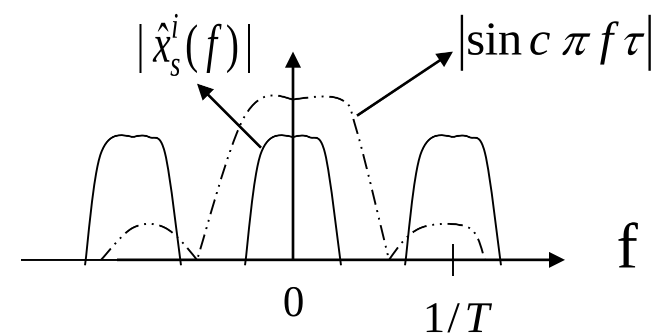
$$\hat{x}_k = \sum_{n=0}^{N-1} x_n . e^{-\frac{2j\pi kn}{N}}$$

Sample and hold







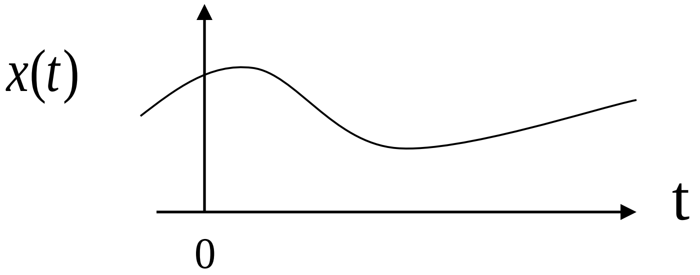


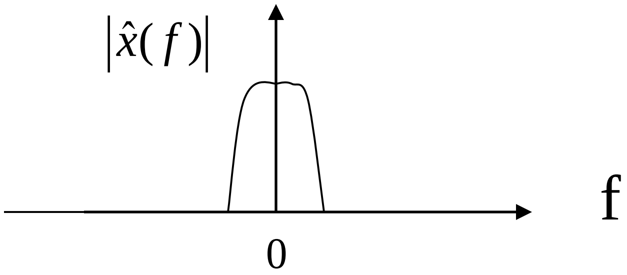
$$x_s(t) = S_a \frac{(t-\tau/2)}{[\tau]} * \sum_{-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

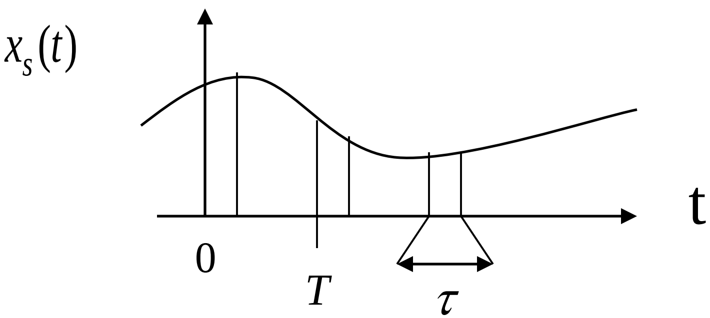
$$\hat{x}_s^r(f) = \tau \sin c\, \pi f \tau . e^{-j\pi f \tau} . \underbrace{\hat{x}(f) * \frac{1}{T}\sum_{-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right)}_{\hat{x}_S^i(f)=\frac{1}{T}\sum_{-\infty}^{\infty}\hat{x}\left(f-\frac{n}{T}\right)}$$

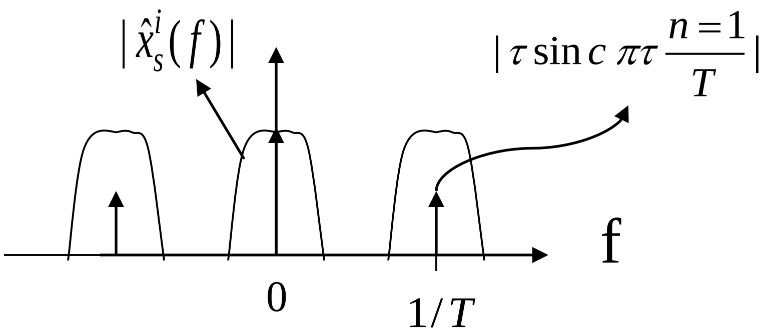
$$\hat{x}_S^i(f) = ideal$$

Average sampling









$$x_s(t) = x(t).\left[S_a \frac{(t-\tau/2)}{[\tau]} * \sum_{-\infty}^{\infty} \delta(t-nT) \right]$$

$$\hat{x}_s^r(f) = \underbrace{\frac{1}{T}\sum_{-\infty}^{\infty}\hat{x}(f-\frac{n}{T})}_{\hat{x}_S^i(f)}.\tau \sin c\, \pi\tau\frac{n}{T}.e^{-j\pi\tau\frac{n}{T}}$$

FFT principle

- Decomposition in odd and even index

$$\hat{x}_k = \sum_{n=0}^{N-1} x_n.e^{-\frac{2j\pi kn}{N}} = \sum_{n=0}^{N-1} x_n.W_N^{nk} \qquad with \quad W_N = e^{-\frac{2j\pi}{N}}$$

$$\hat{x}_k = \sum_{n=0}^{\frac{N}{2}-1} x_{2n}.W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1}.W_N^{(2n+1)k}$$

Suite of decomposition

$$\hat{x}_k = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_{2n}.W_{\frac{N}{2}}^{nk}}_{G_K} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_{2n+1}.W_{\frac{N}{2}}^{nk}}_{H_K}$$

suite

$$\hat{x}_k = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_{2n}.W_{\frac{N}{2}}^{nk}}_{G_K} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_{2n+1}.W_{\frac{N}{2}}^{nk}}_{H_K}$$

- Gk got with a DFT on the even index
- Hk got with a DFT on the odd index

First conclusion

- Thus the computaion of the DFT on N points can be get with the 2 DFT on N/2 points.
- Also we note that :

$$G_{k+\frac{N}{2}} = \sum_{n=0}^{\frac{N}{2}-1} x_{2n}.W_{\frac{N}{2}}^{n(k+\frac{N}{2})} = \sum_{n=0}^{\frac{N}{2}-1} x_{2n}.W_{\frac{N}{2}}^{n(k+\frac{N}{2})} \qquad and \quad W_{\frac{N}{2}}^{\frac{nN}{2}} = e^{-j2\pi n} = 1$$

$$G_{k+\frac{N}{2}} = \sum_{n=0}^{\frac{N}{2}-1} x_{2n}.W_{\frac{N}{2}}^{nk} = G_k \qquad and \quad W_N^{k+\frac{N}{2}} = e^{\frac{-j2\pi}{N}\left(k+\frac{N}{2}\right)} = -W_N^k$$

$$H_{k+\frac{N}{2}} = H_k$$

Assessement

$$\left.\begin{aligned}\hat{x}_k &= G_k + W_N^k H_k \\ \hat{x}_{k+\frac{N}{2}} &= G_k - W_N^k H_k\end{aligned}\right\} \quad k = \left[0, \frac{N}{2} - 1\right]$$

- Knowing Gk and Hk for $k = \left[0, \frac{N}{2} - 1\right]$
- We use N/2 complexes multiplications and

  N complexes additions

Assessement (suite)

We can continue

$$\left\{\begin{array}{l}\left.\begin{array}{l}G_k = A_k + W_{\frac{N}{2}}^{k} B_k \\ G_{k+\frac{N}{4}} = A_k - W_{\frac{N}{2}}^{k} B_k\end{array}\right\} \quad k \in \left[0, \frac{N}{4} - 1\right] \\ \\ \left.\begin{array}{l}H_k = C_k + W_{\frac{N}{2}}^{k} D_k \\ H_{k+\frac{N}{4}} = C_k - W_{\frac{N}{2}}^{k} D_k\end{array}\right\} \quad k \in \left[0, \frac{N}{4} - 1\right]\end{array}\right.$$

Second conclusion (cut out)

- Knowing Ak and Bk, for $k \in \left[0, \frac{N}{4} - 1\right]$

  We use N/4 complexes multiplications and
  N/2 complexes additions

- Knowing Ck and Dk, for $k \in \left[0, \frac{N}{4} - 1\right]$

  We use N/4 complexes multiplications and
  N/2 complexes additions

That is to say : N/2 complexes multiplications and
N complexes additions

Assessement

- Each step use N/2 complexes multiplications and N complexes additions

- With $N = 2^{\gamma}$ $\gamma = number\ of\ steps$
- We will use

$$\frac{N}{2}\gamma = \frac{N}{2}.Log_2 N \quad complexes\ multiplications$$

$$N\gamma = N.Log_2 N \quad complexes\ additions$$

FFT Principle (for N=8)

$$\begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \vdots \\ \hat{x}_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \cdots & 1 \\ 1 & W_8^1 \cdots W_8^7 \\ \vdots & & \\ 1 & W_8^7 \cdots W_8^{49} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_7 \end{bmatrix}$$

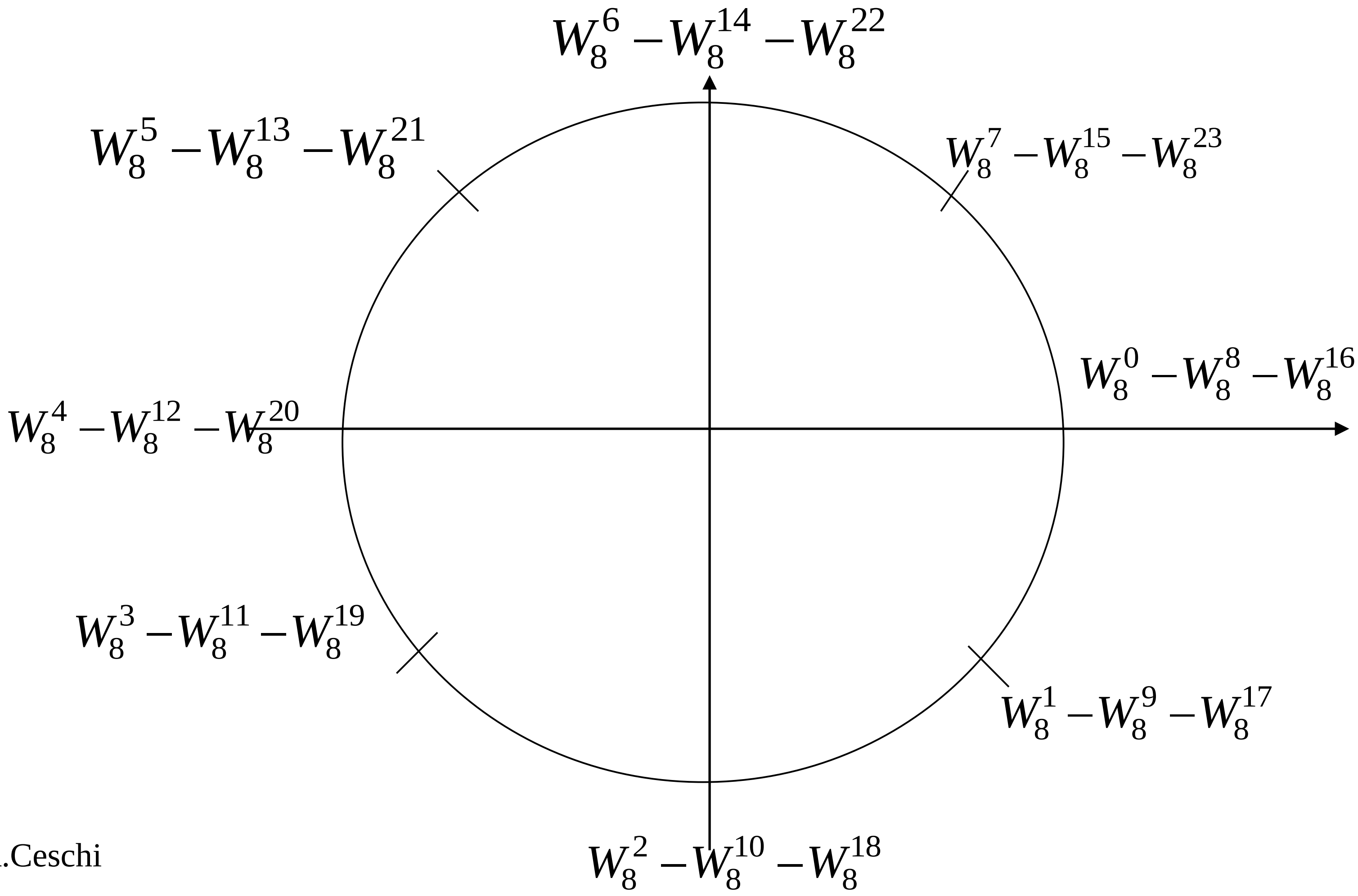
which can be cut out

Decomposition

$$\begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_8^2 & W_8^4 & W_8^6 \\ 1 & W_8^4 & W_8^8 & W_8^{12} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ W_8^1 & W_8^3 & W_8^5 & W_8^7 \\ W_8^2 & W_8^6 & W_8^{10} & W_8^{14} \\ W_8^3 & W_8^9 & W_8^{15} & W_8^{21} \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}_4 \\ \hat{x}_5 \\ \hat{x}_6 \\ \hat{x}_7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & W_8^8 & W_8^{16} & W_8^{24} \\ 1 & W_8^{10} & W_8^{20} & W_8^{30} \\ 1 & W_8^{12} & W_8^{24} & W_8^{36} \\ 1 & W_8^{14} & W_8^{28} & W_8^{42} \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} + \underbrace{\begin{bmatrix} W_8^4 & W_8^{12} & W_8^{20} & W_8^{28} \\ W_8^5 & W_8^{15} & W_8^{25} & W_8^{35} \\ W_8^6 & W_8^{18} & W_8^{30} & W_8^{42} \\ W_8^7 & W_8^{21} & W_8^{35} & W_8^{49} \end{bmatrix}}_{\mathcal{D}} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

With $W_N = e^{-j\frac{2\pi}{N}}$ *with* $N = 8$



Simplification

- We note that :

$$\mathcal{A} = \mathcal{C} \quad and \quad \mathcal{B} = -\mathcal{D} \quad also$$

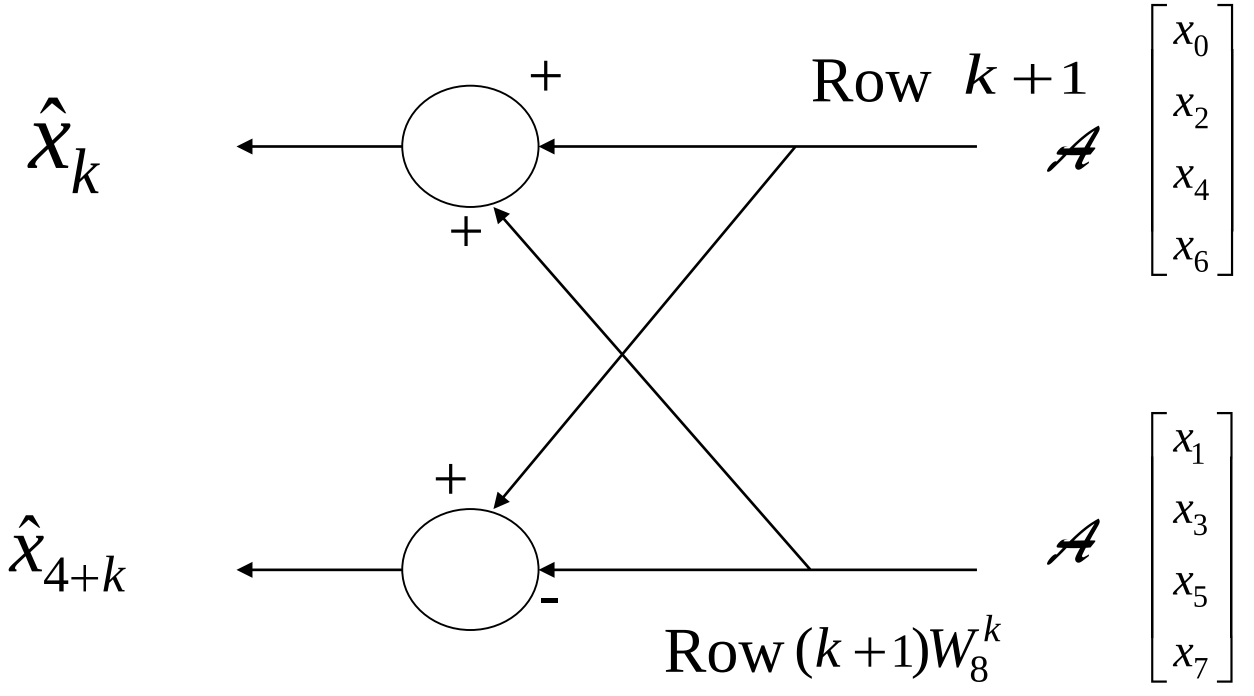
$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \mathcal{A}$$

thus

$$\begin{bmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathcal{A}\begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \mathcal{A}\begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}_4 \\ \hat{x}_5 \\ \hat{x}_6 \\ \hat{x}_7 \end{bmatrix} = \mathcal{A}\begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \mathcal{A}\begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

From where



$\hat{x}_k$
$\hat{x}_{4+k}$
+
+
+
-
Row $k+1$
Row $(k+1)W_8^k$
$\mathcal{A}$
$\mathcal{A}$
x_0
x_2
x_4
x_6
x_1
x_3
x_5
x_7

Continue...

- But for computing

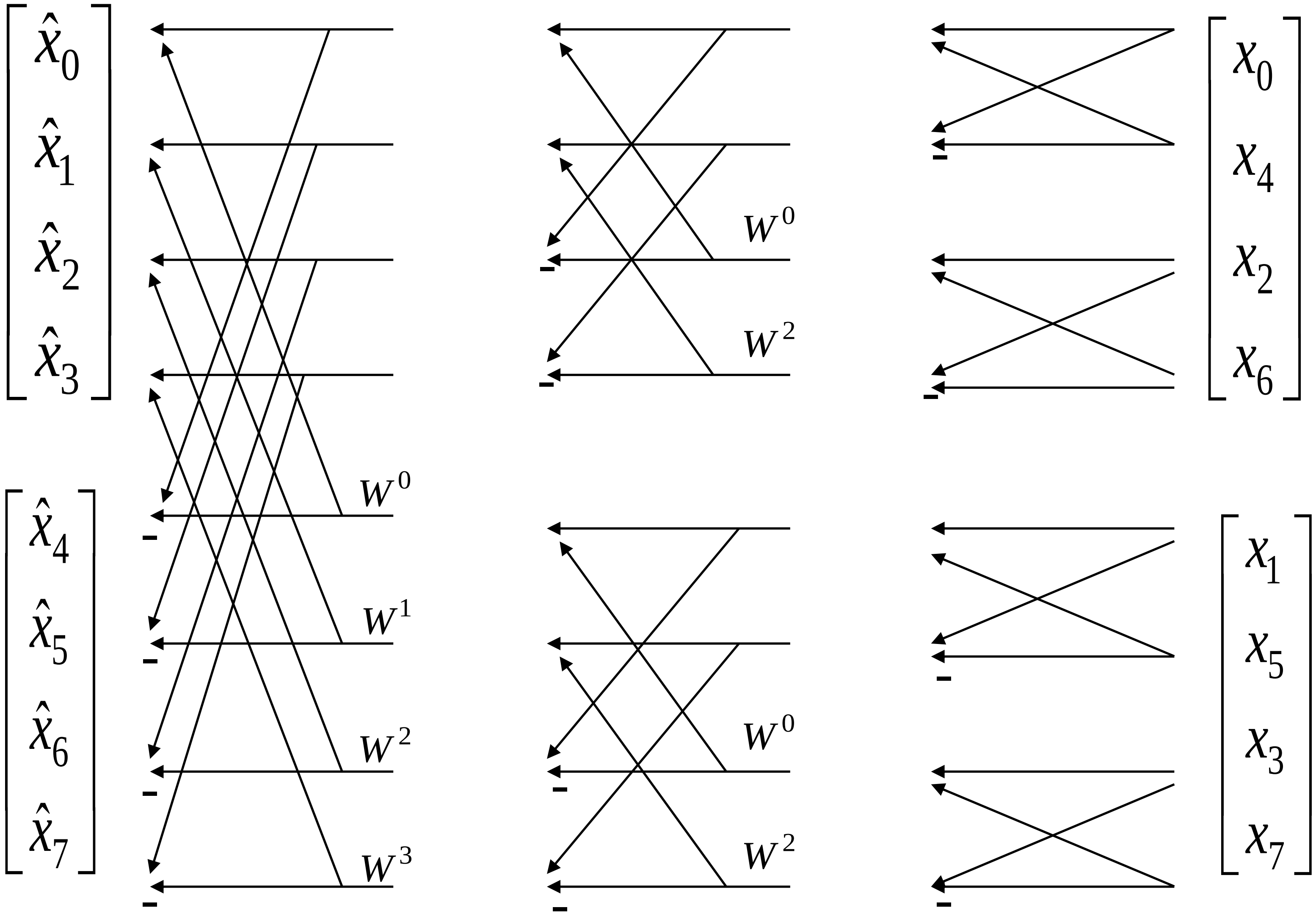
$$\mathcal{A}\begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} \quad or \quad \mathcal{A}\begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{bmatrix}$$

- We can use the same decomposition

Let us develop

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_8^2 & W_8^4 & W_8^6 \\ 1 & W_8^4 & W_8^8 & W_8^{12} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ x_6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & W_8^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_6 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & W_8^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_6 \end{bmatrix} \end{bmatrix}$$

$\mathcal{A}$



$\hat{x}_0$
$\hat{x}_1$
$\hat{x}_2$
$\hat{x}_3$
$\hat{x}_4$
$\hat{x}_5$
$\hat{x}_6$
$\hat{x}_7$
W^0
W^1
W^2
W^3
W^0
W^2
W^0
W^2
-
x_0
x_4
x_2
x_6
x_1
x_5
x_3
x_7

End of algorithme

- The last step of the algorithme : inverse the bits order in the input sequence.

000	000
001	100
010	010
011	110

$$input\ sequence\ "abc" \rightarrow "cba" \quad output\ sequence$$

100	001
101	101
110	011
111	111

Bibliographie

- http://ocw.mit.edu/NR/rdonlyres/Mechanical-Engineering/2-161Fall-2008/34C75681-5793-4DCF-9D66-7F9861E38CDA/0/fft.pdf

Have a good day