State Space Analysis: Properties, Reachability Graph, and Coverability graph

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Where innovation starts

Outline

- Motivation
- Formalization
- Basic properties
- Reachability graph
- Coverability graph

Motivation

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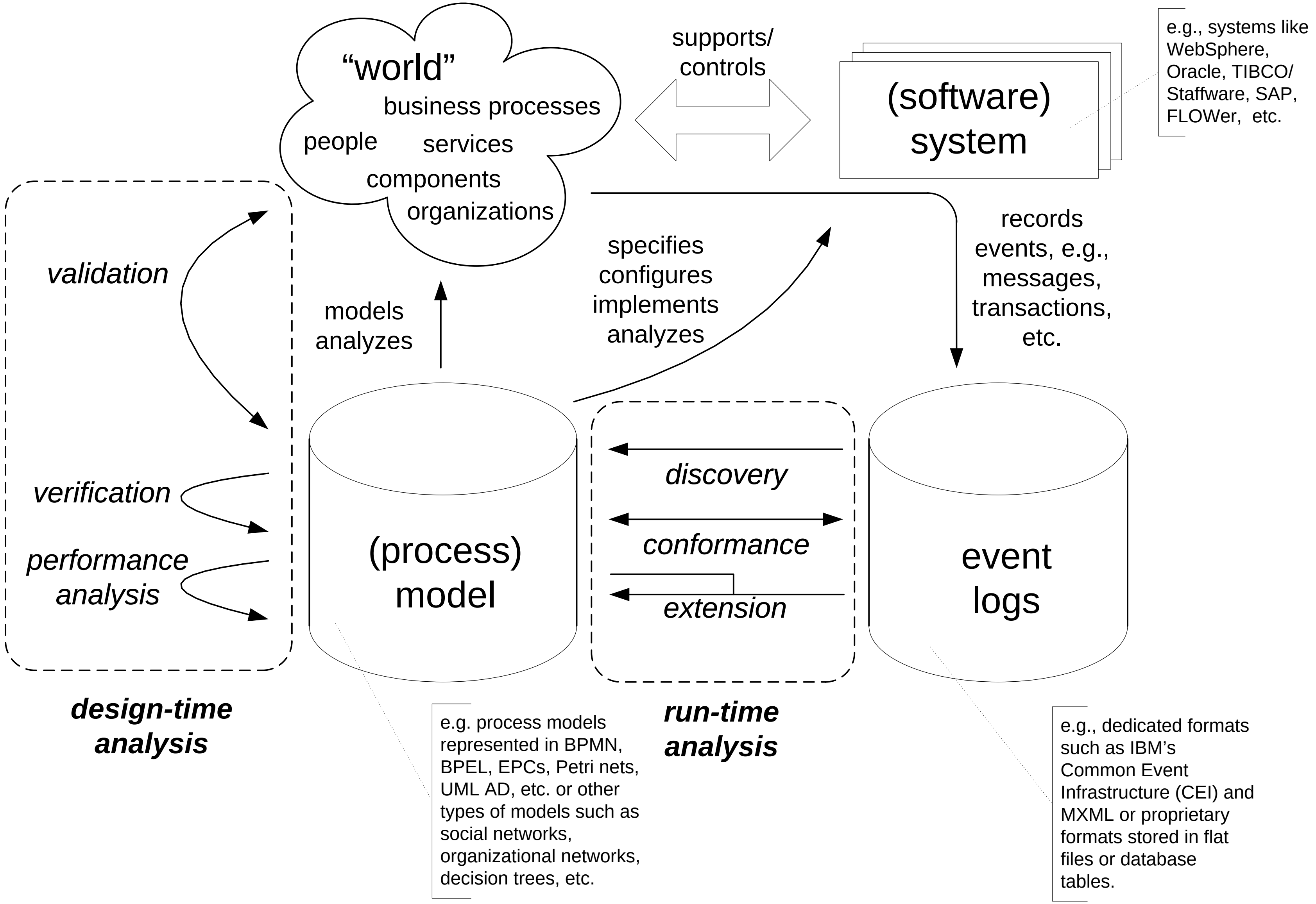
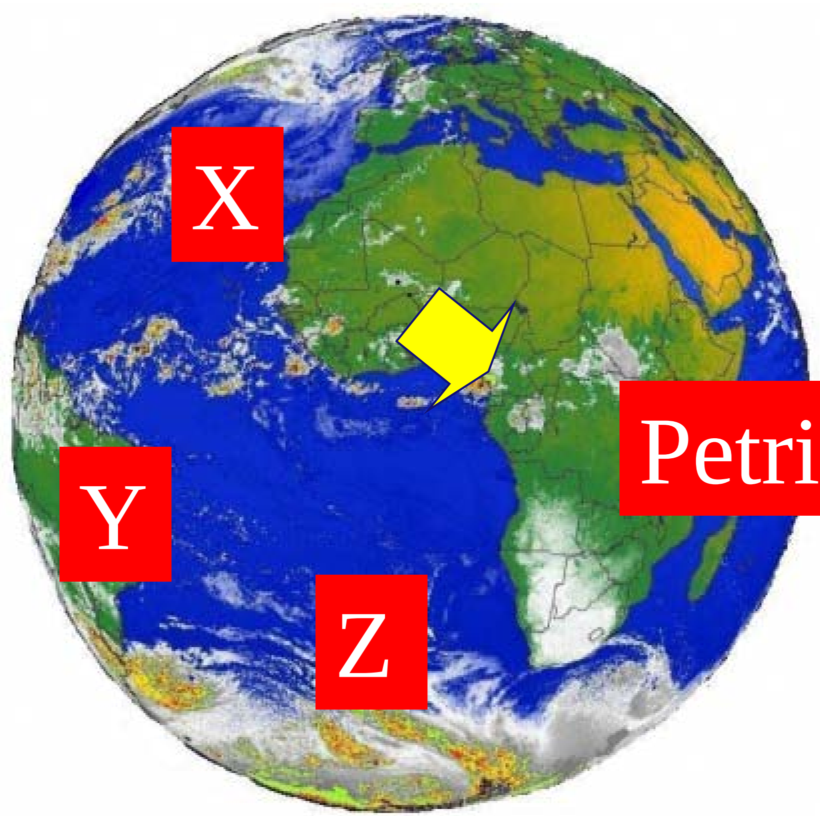
Design-time analysis vs run-time analysis



Analysis of processes



Petri net

linear algebraic analysis techniques

Markov chain analysis techniques

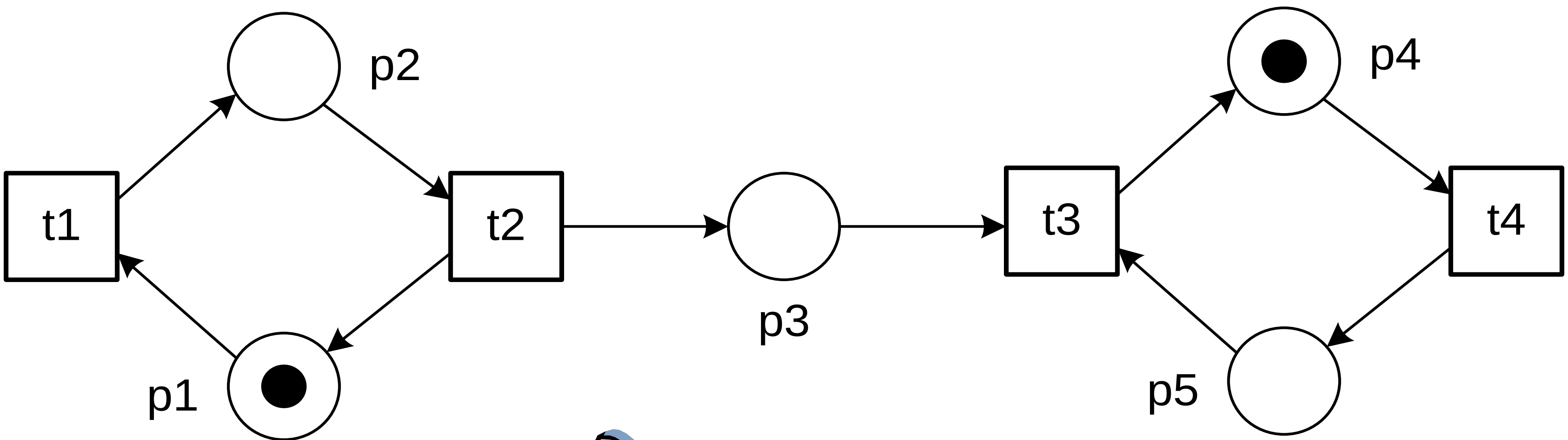
state-space analysis techniques

....

Generic questions

terminating
it has only finite occurrence sequences
deadlock-free
each reachable marking enables a transition
live
each reachable marking enables an occurrence sequence containing all transitions
bounded
each place has an upper bound that holds for all reachable markings
1-safe
1 is a bound for each place s
reversible
m0 is reachable from each reachable marking, i.e., the initial marking is a so-called **home marking**.

Example



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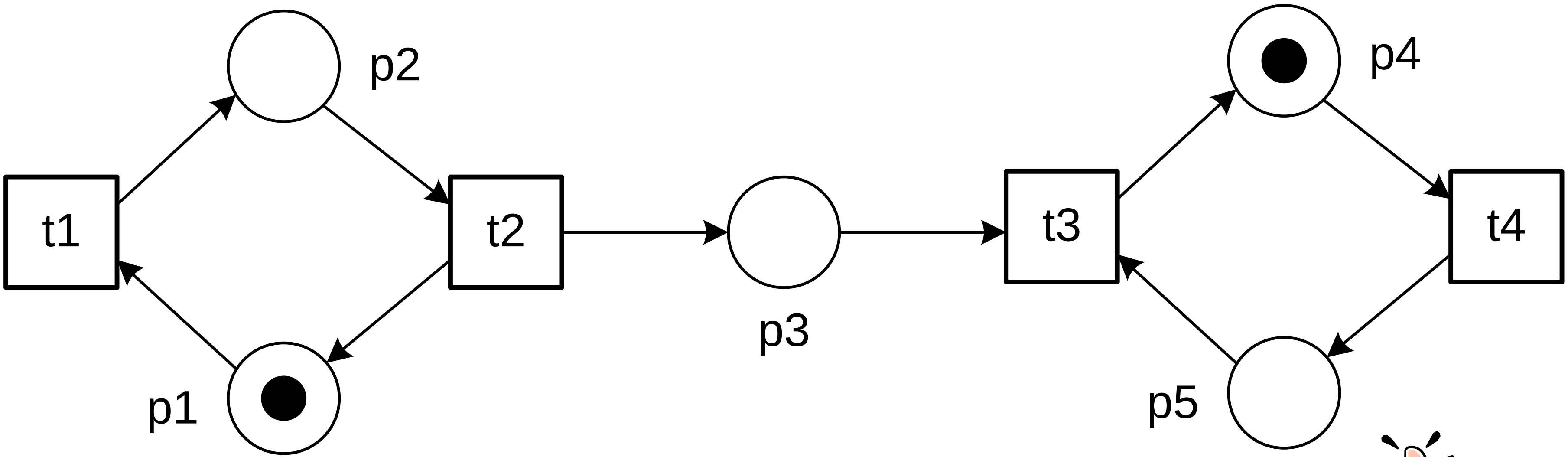
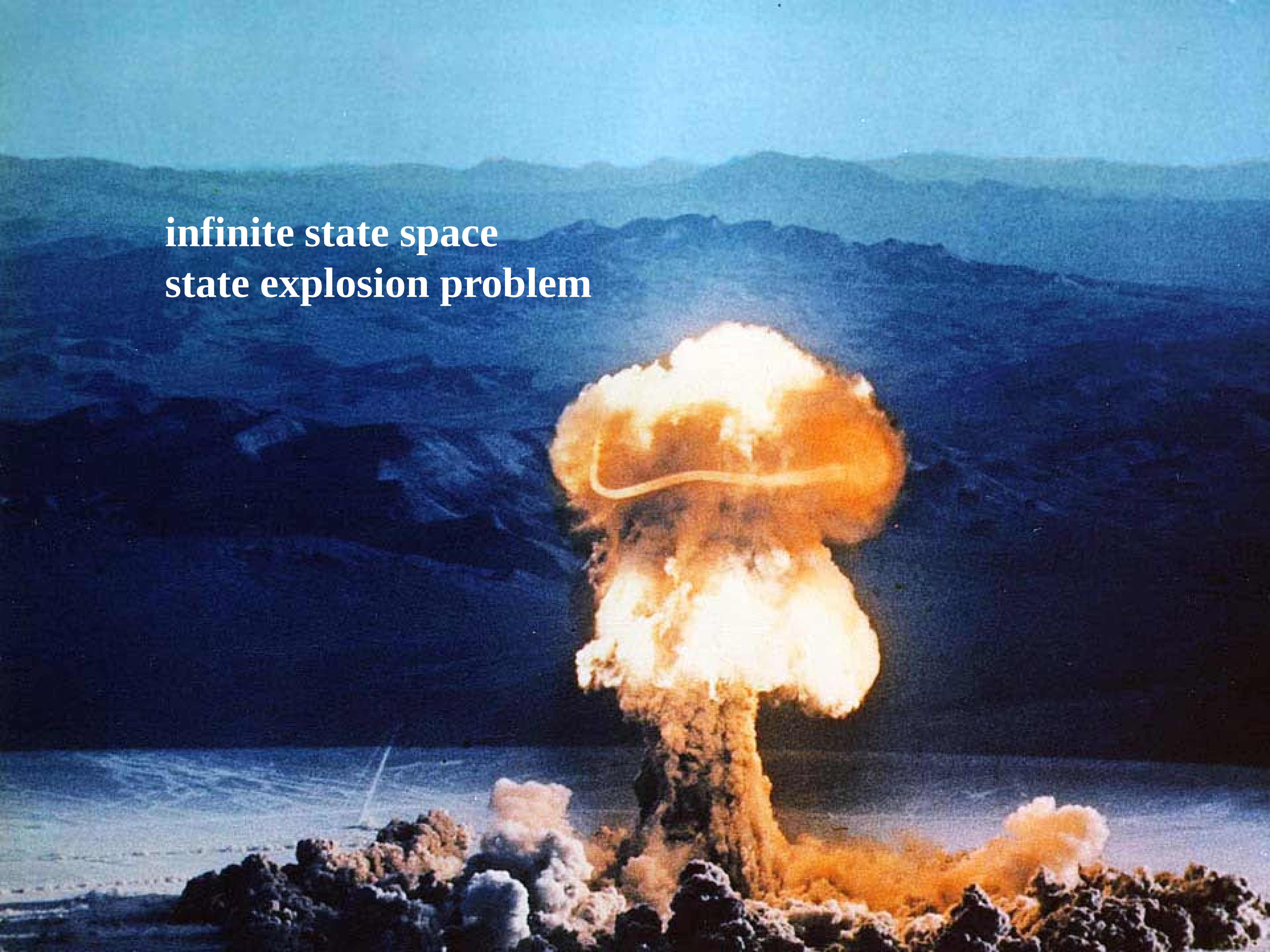
Specific questions

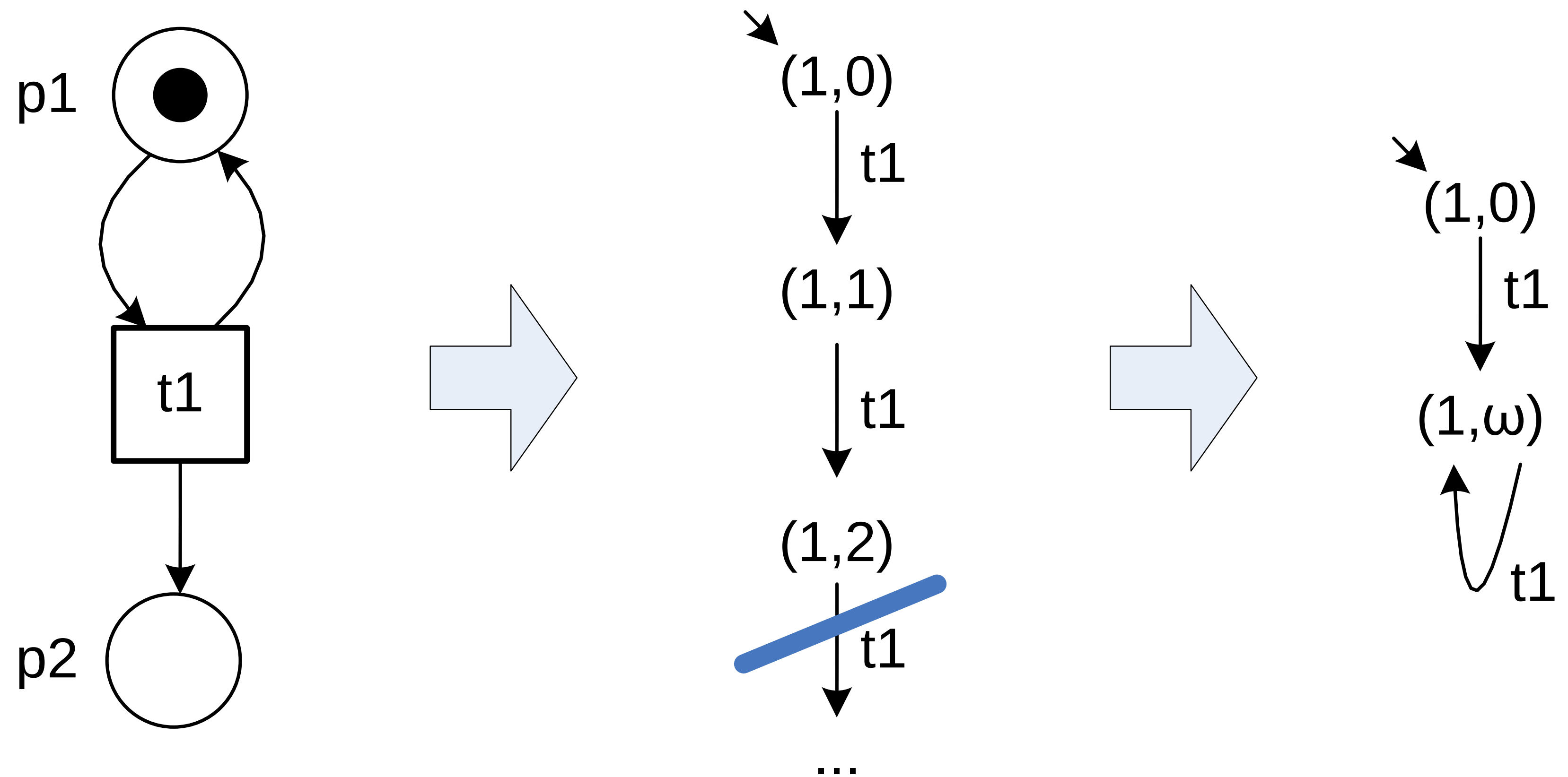


Is it possible to have a token in both p2 and p5?
Will t3 always take place?
Will t3 always take place assuming "fairness"?
Is it possible to execute t1 after t4?
Can both p4 and p5 be empty at the same time?



infinite state space
state explosion problem

Concepts



marked net *reachability graph* *coverability graph*

Relevant material

1. **Jörg Desel, Wolfgang Reisig: Place/Transition Petri Nets. Petri Nets 1996: 122-173.** DOI: 10.1007/3-540-65306-6_15
http://www.springerlink.com/content/x6hn592l35866lu8/fulltext.pdf
2. **Tadao Murata, Petri Nets: Properties, Analysis and Applications, Proceedings of the IEEE. 77(4): 541-580, April, 1989.** http://dx.doi.org/10.1109/5.24143
http://ieeexplore.ieee.org/iel1/5/911/00024143.pdf
3. **Wil van der Aalst: Process Mining: Discovery, Conformance and Enhancement of Business Processes, Springer Verlag 2011 (chapters 1 & 5)**
   a) **Chapter 1:** DOI: 10.1007/978-3-642-19345-3_1
   http://www.springerlink.com/content/p443h219v3u3537l/fulltext.pdf
   b) **Chapter 5:** DOI: 10.1007/978-3-642-19345-3_5
   http://www.springerlink.com/content/u58h17n3167p0x1u/fulltext.pdf
   c) **Events logs:** http://www.processmining.org/book/

Today's focus is on 1 & 2.

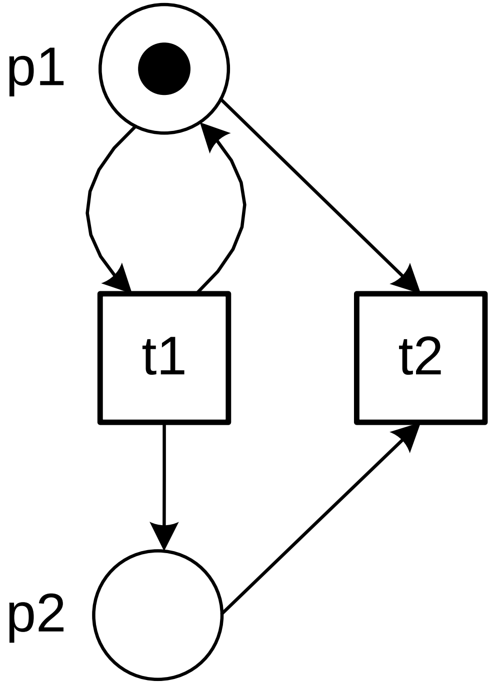
Formalization

Note: refinement of earlier link between Petri net and transitions system (week 2/3) that is closer to standard literature.

Basic Petri net

Definition 1 (Basic Petri net). *A basic Petri net is a triple (P, T, F). P is a finite set of places, T is a finite set of transitions ($P \cap T = \emptyset$), and $F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs (flow relation).*

- **P = {p1,p2}**
- **T = {t1,t2}**
- **F = {(p1,t1), (t1,p1), (t1,p2), (p1,t2), (p2,t2)}**

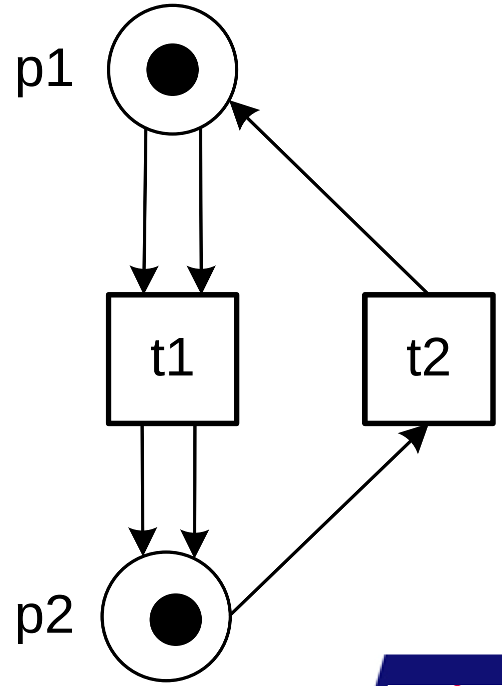


Place transition net

Definition 2 (Place transition net (PT-net)). *An Place transition net (or simply Petri net) is a tuple (P, T, F, W), where:*

- *(P, T, F) is a basic Petri net,*
- *$W \in F \to \mathbb{N} \setminus \{0\}$ is an (arc) weight function.*

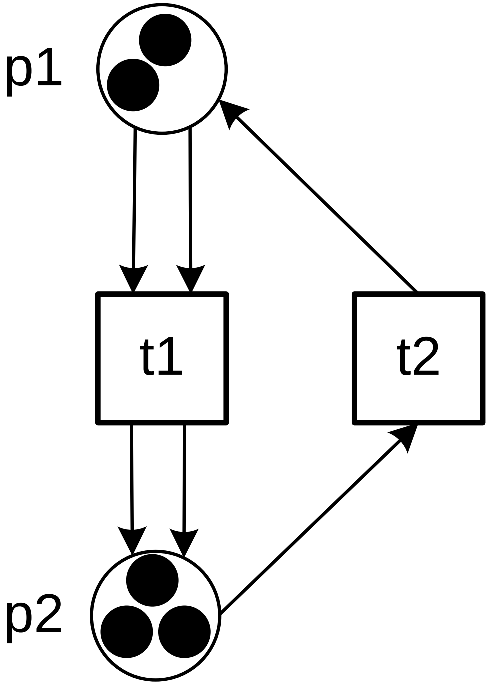
- **P = {p1,p2}**
- **T = {t1,t2}**
- **F = {(p1,t1), (t1,p2), (p2,t2), (t2,p1)}**
- **W(p1,t1)=2, W(t1,p2)=2, W(p2,t2)=1, and W(t2,p1)=1**



Multi-sets

Definition 3 (Multi-set). *Let A be a set. $\mathbb{B}(A) = A \rightarrow \mathbb{N}$ is the set of multi-sets (bags) over A, i.e., $X \in \mathbb{B}(A)$ is a multi-set where for each $a \in A$: $X(a)$ denotes the number of times a is included in the multi-set.*

- **$M_0(p1) = 2$**
- **$M_0(p2) = 3$**



Operations on multi-sets

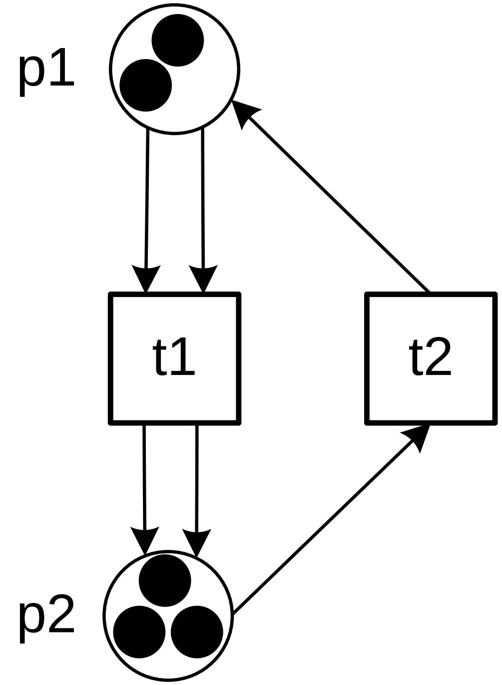
Let X and Y be two multi-sets

- The sum of two multi-sets $(X+Y)$, the difference $(X-Y)$, the presence of an element in a multi-set $(x \in X)$, and the notion of sub-multi-set $(X \leq Y)$ are defined in a straightforward way.
- They can handle a mixture of sets and multi-sets.
- The operators are also robust with respect to the domains of the multi-sets, i.e., even if X and Y are defined on different domains, $X+Y$, $X-Y$, and $X \leq Y$ are defined properly by taking the union of the domains where needed.
- $|X| = \sum_{a \in A} X(a)$ is the size of some multi-set X over A.
- $X(A') = \sum_{a \in A'} X(a)$ denotes the number of elements in X with a value in $A' \subseteq A$.
- $\pi_{A'}(X)$ is the projection of X onto $A' \subseteq A$, i.e., $(\pi_{A'}(X))(a) = X(a)$ if $a \in A'$ and $(\pi_{A'}(X))(a) = 0$ if $a \notin A'$.

Notation

To represent a concrete multi-set we use square brackets, e.g., $[a, a, b, a, b, c]$, $[a^3, b^2, c]$, and $3[a] + 2[b] + [c]$ all refer to the same multi-set with six elements: 3 a's, 2 b's, and one c. $[\,]$ refers to the empty bag, i.e., $|[\,]| = 0$.

- **M_0 = [p1,p1,p2,p2,p2] = [$p1^2$,$p2^3$] = 2[p1]+3[p2]**
- **also denoted as (2,3)**



Preset/postset

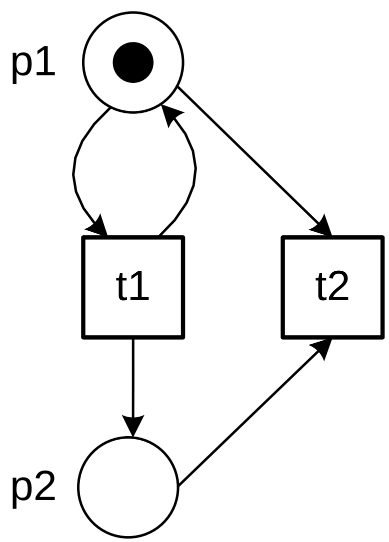
Definition 4 (Marking). *Let* $N = (P, T, F, W)$ *be a Petri net. A marking* M *of* N *is a multi-set over* P*, i.e.,* $M \in \mathbb{B}(P)$.

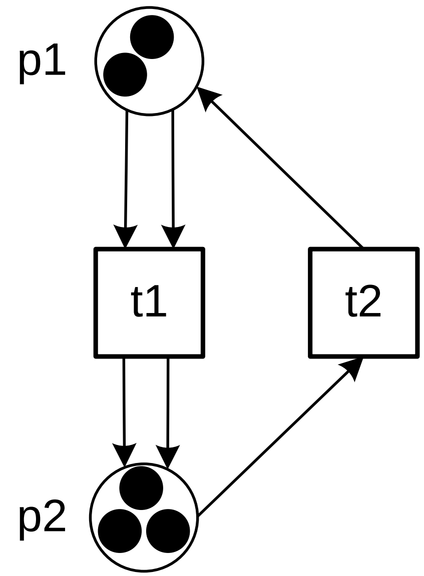
Definition 5 (Preset,postset). *Let* $N = (P, T, F, W)$ *be a Petri net.*

- $\bullet a = [x^{W(x,y)} \mid (x, y) \in F \;\wedge\; a = y]$ *is the preset of* a.
- $a \bullet = [y^{W(x,y)} \mid (x, y) \in F \;\wedge\; a = x]$ *is the postset of* a.
- *Moreover, we extend the weight function for the situation that there is not an arc connecting two nodes, i.e.,* $W(x, y) = 0$ *if* $(x, y) \notin F$.

Examples

- •p1 = [t1]
- p1• = [t1,t2]
- •p2 = [t1]
- p2• = [t2]
- •t1 = [p1]
- t1• = [p1,p2]
- •t2 = [p1,p2]
- t2• = [ ]



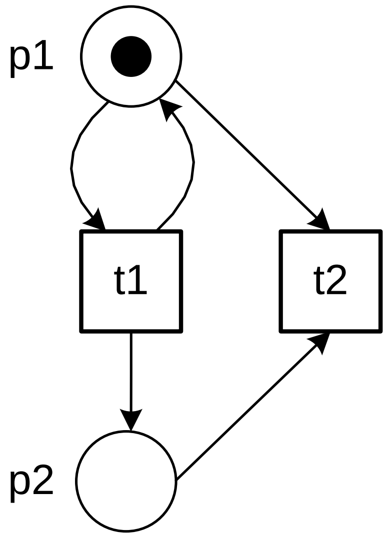


- •p1 = [t2]
- p1• = [$t1^2$]
- •p2 = [$t1^2$]
- p2• = [t2]
- •t1 = [$p1^2$]
- t1• = [$p2^2$]
- •t2 = [p2]
- t2• = [p1]

Firing rule

Definition 6 (Firing rule). *Let* $N = (P, T, F, W)$ *be a Petri net and* $M \in \mathbb{B}(P)$ *be a marking.*

- *A transition* $t \in T$ *is enabled, notation* $(N, M)[t\rangle$*, if and only if,* $M \geq \bullet t$.
- *An enabled transition* t *can fire while changing the state to* M'*, notation* $(N, M)[t\rangle(N, M')$*, if and only if,* $M' = (M - \bullet t) + t\bullet$.



Notations

Table 2 Formal Definition of a Petri Net

A Petri net is a 5-tuple, $PN = (P, T, F, W, M_0)$ where:

$P = \{p_1, p_2, \cdots, p_m\}$ is a finite set of places,
$T = \{t_1, t_2, \cdots, t_n\}$ is a finite set of transitions,
$F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs (flow relation),
$W: F \rightarrow \{1, 2, 3, \cdots\}$ is a weight function,
$M_0: P \rightarrow \{0, 1, 2, 3, \cdots\}$ is the initial marking,
$P \cap T = \varnothing$ and $P \cup T \neq \varnothing$.

A Petri net structure $N = (P, T, F, W)$ without any specific initial marking is denoted by N.

A Petri net with the given initial marking is denoted by (N, M_0).

Murata

A *net* N is constituted by

- a set S of *places*,
- a set T of *transitions* such that $S \cap T = \emptyset$, and
- a set F of *directed arcs* (*flow relation*), $F \subseteq (S \cup T) \times (S \cup T)$, satisfying

$$F \cap (S \times S) = F \cap (T \times T) = \emptyset.$$

Desel/Reisig

Notations: Firing rule

The behavior of many systems can be described in terms of system states and their changes. In order to simulate the dynamic behavior of a system, a state or marking in a Petri nets is changed according to the following *transition* (*firing*) *rule*:

1) A transition t is said to be *enabled* if each input place p of t is marked with at least $w(p, t)$ tokens, where $w(p, t)$ is the weight of the arc from p to t.
2) An enabled transition may or may not fire (depending on whether or not the event actually takes place).
3) A firing of an enabled transition t removes $w(p, t)$ tokens from each input place p of t, and adds $w(t, p)$ tokens to each output place p of t, where $w(t, p)$ is the weight of the arc from t to p.

Murata

A *marking* of a net N is a mapping $m: S_N \rightarrow I\!N$ where $I\!N = \{0, 1, 2, \ldots\}$. A place s is *marked* by a marking m if $m(s) > 0$. The *null marking* is the marking which maps every place to 0.

A transition t is *enabled* by a marking m if m marks all places in ${}^{\bullet}t$. In this case t can *occur*. Its occurrence transforms m into the marking m', defined for each place s by

$$m'(s) = \begin{cases} m(s) - 1 & \text{if } s \in {}^{\bullet}t - t^{\bullet}, \\ m(s) + 1 & \text{if } s \in t^{\bullet} - {}^{\bullet}t, \\ m(s) & \text{otherwise.} \end{cases}$$

Desel/Reisig

Basic Properties

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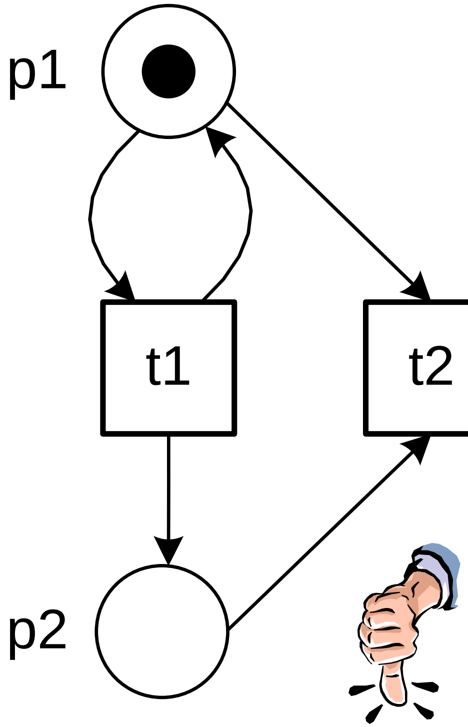
Basic properties of a marked Petri net

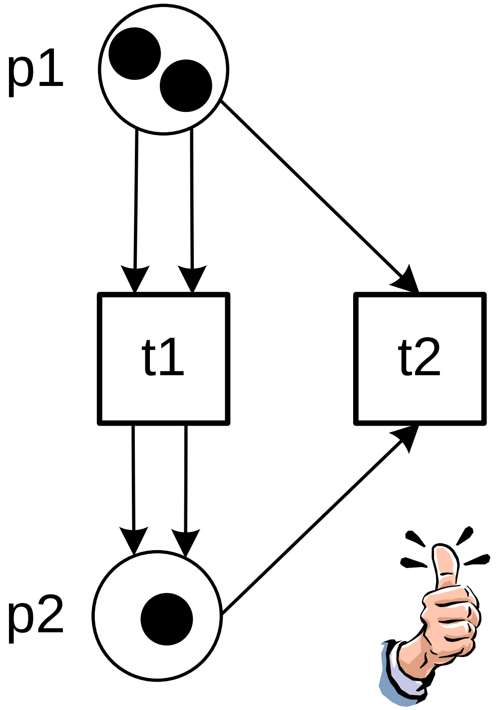
Definition 9 (Basic properties). *Let $N = (P, T, F, W)$ be a Petri net and $M \in \mathbb{B}(P)$ be a marking.*

- (N, M) *is* terminating *if and only if there is a* $k \in \mathbb{N}$ *such that* $|\sigma| \leq k$ *for any firing sequence* σ *(i.e.,* $(N, M)[\sigma\rangle$*).*
- (N, M) *is* deadlock-free *if and only if for any* $M' \in R(N, M)$ *there exists a transition* t *such that* $(N, M')[t\rangle$.
- (N, M) *is* live *if and only if for any* $t \in T$ *and any* $M' \in R(N, M)$ *there exists a* $M'' \in R(N, M')$ *such that* $(N, M'')[t\rangle$.
- (N, M) *is* bounded *if and only if there is a* $k \in \mathbb{N}$ *such that for any* $M' \in R(N, M)$ *and any* $p \in P$*:* $M'(p) \leq k$.
- (N, M) *is* safe *if and only if for any* $M' \in R(N, M)$ *and any* $p \in P$*:* $M'(p) \leq 1$.
- (N, M) *is* reversible *if and only if for any* $M' \in R(N, M)$*:* $M \in R(N, M')$.

Terminating

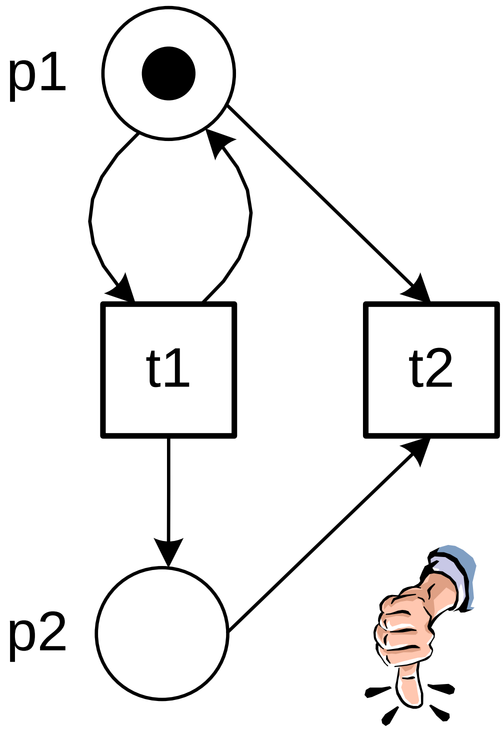
(N, M) is *terminating* if and only if there is a $k \in \mathbb{N}$ such that $|\sigma| \leq k$ for any firing sequence σ (i.e., $(N, M)[\sigma\rangle$).

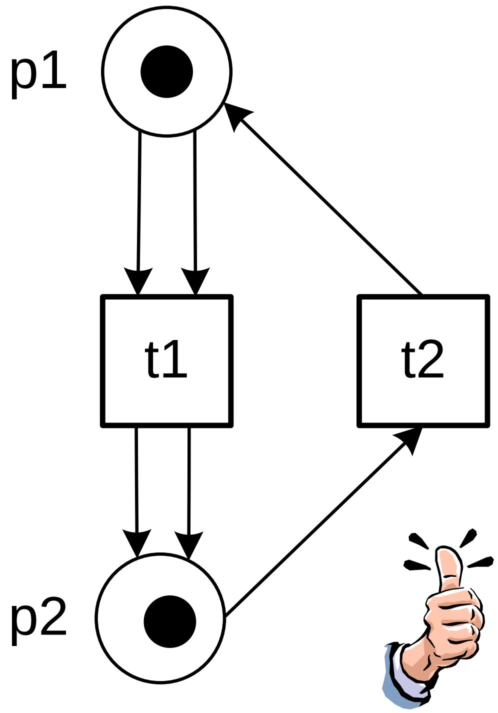




Deadlock-free

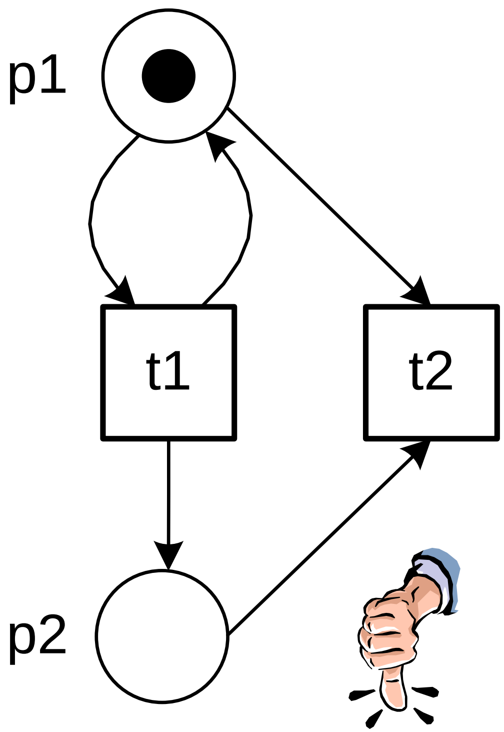
(N, M) is *deadlock-free* if and only if for any $M' \in R(N, M)$ there exists a transition t such that $(N, M')[t\rangle$.

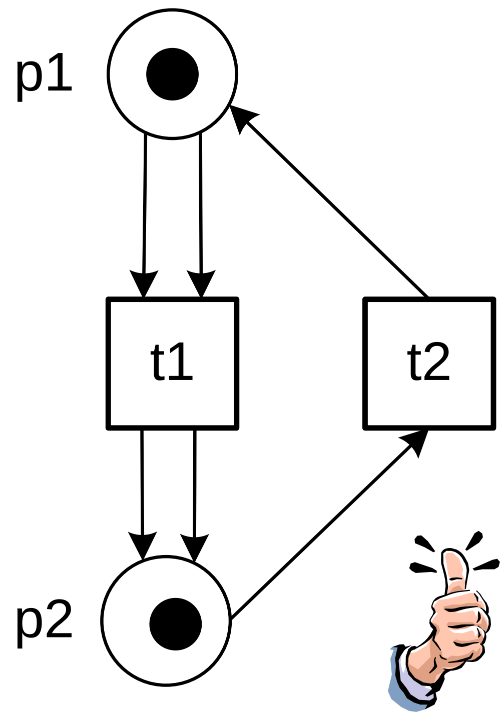




Liveness

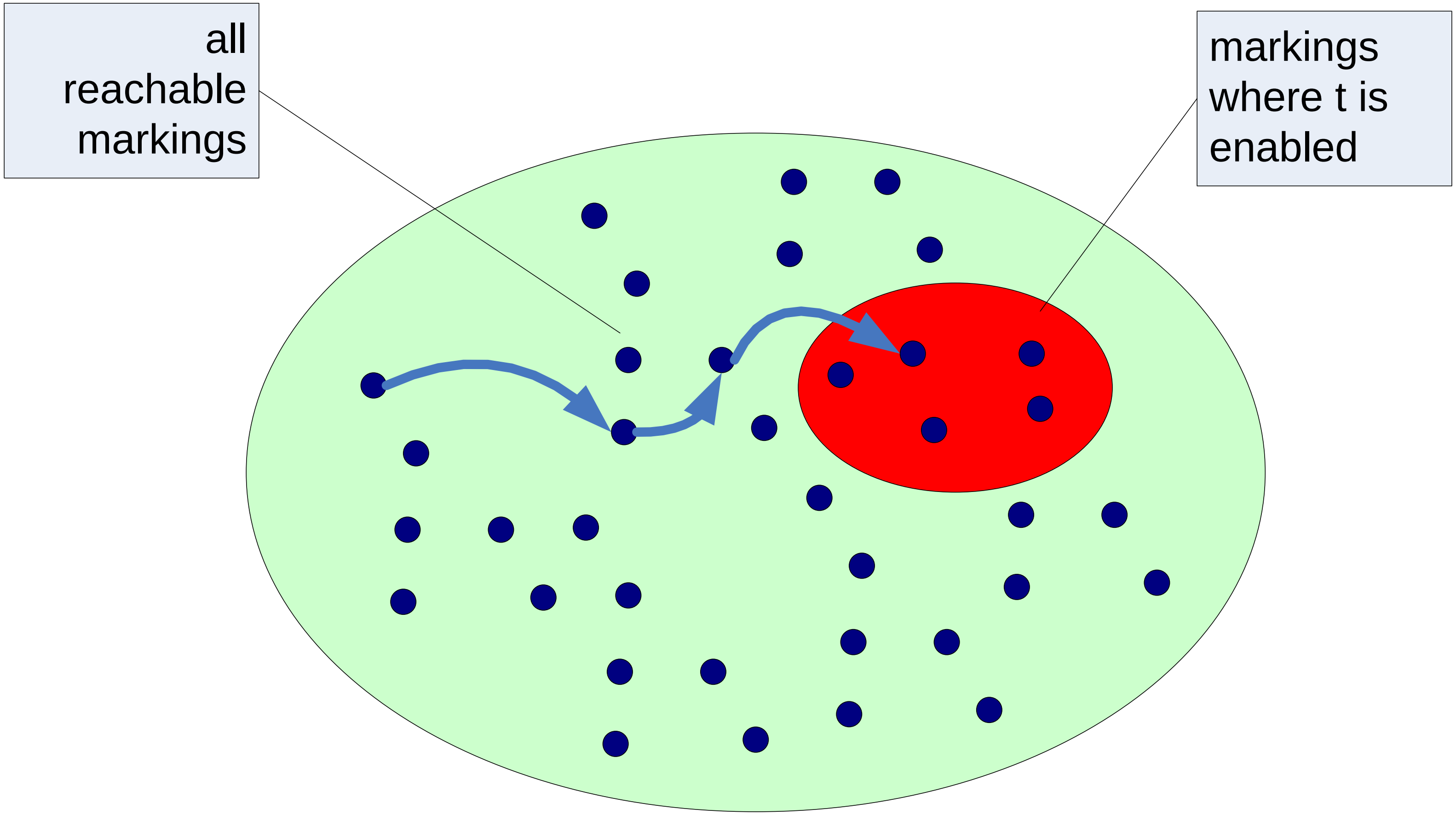
Transition $t \in T$ is *live* in (N, M) if and only if for any $M' \in R(N, M)$ there exists a $M'' \in R(N, M')$ such that $(N, M'')[t\rangle$.





(N, M) is *live* is all of its transitions are live.

Basic idea of liveness

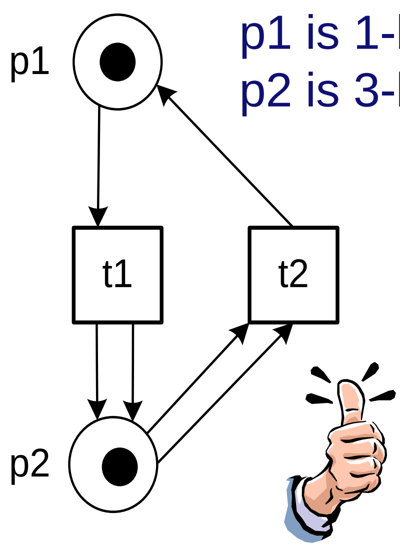


Boundedness

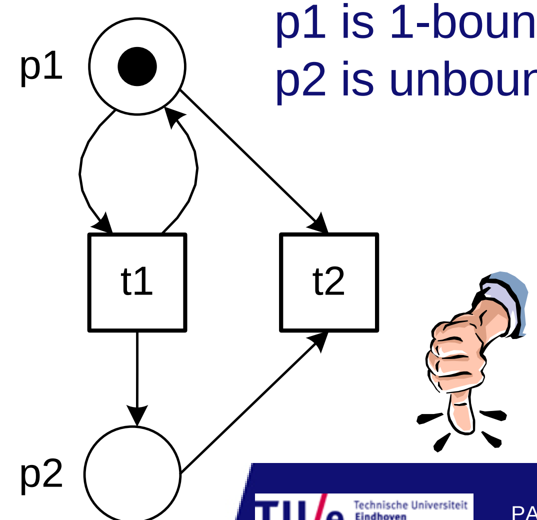
Place $p \in P$ is k-bounded in (N, M) if and only if for any $M' \in R(N, M)$: $M'(p) \leq k$.

Place $p \in P$ is bounded in (N, M) if and only if there is a $k \in \mathbb{N}$ such that p is k-bounded.

(N, M) is *bounded* if and only if all of its places are bounded.



p1 is 1-bounded,
p2 is 3-bounded

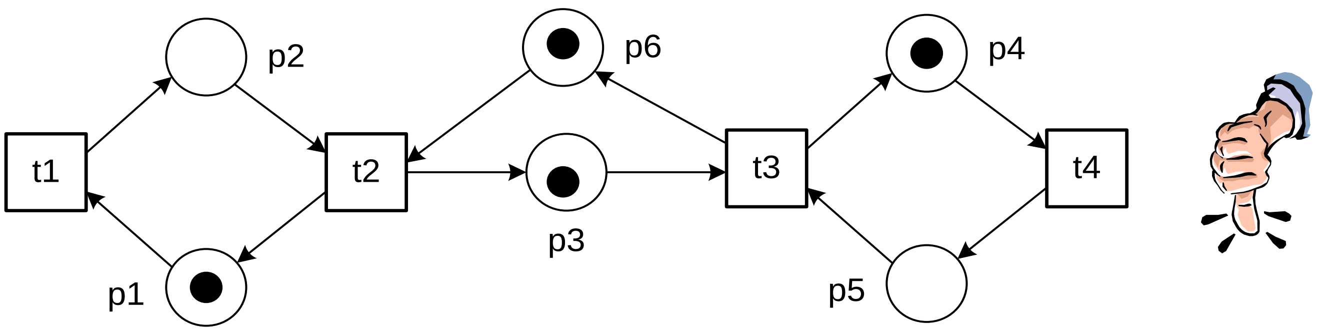


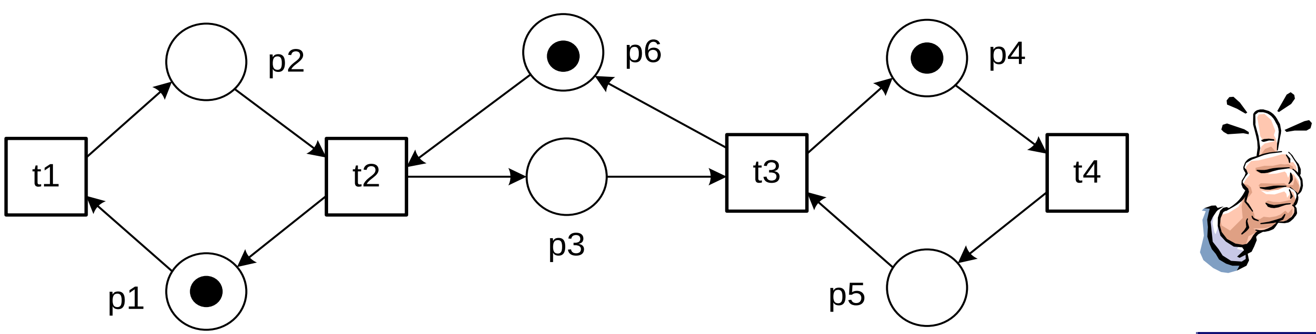
p1 is 1-bounded,
p2 is unbounded

Safeness

Place $p \in P$ is *safe* in (N, M) if and only if p is 1-bounded.

(N, M) is *safe* if and only if all of its places are safe.



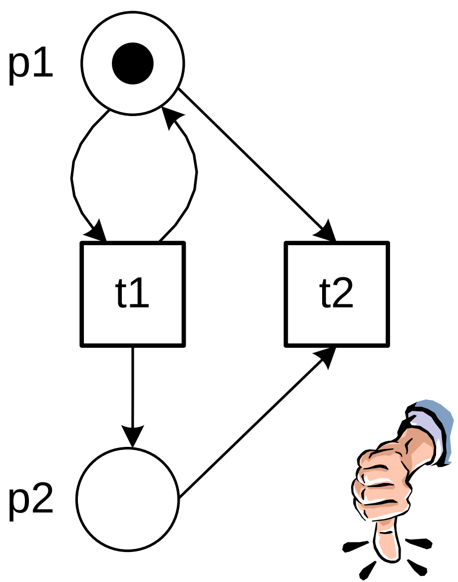


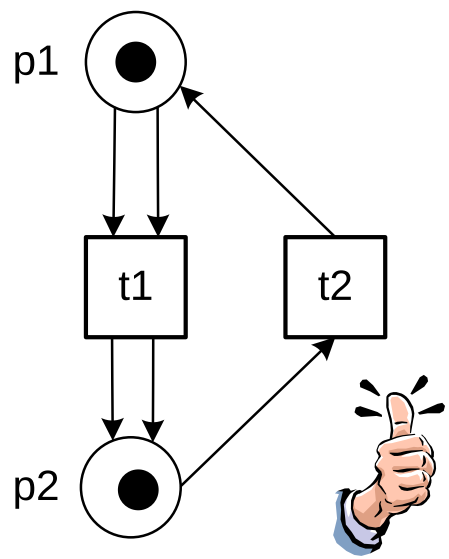
Reversible/home marking.

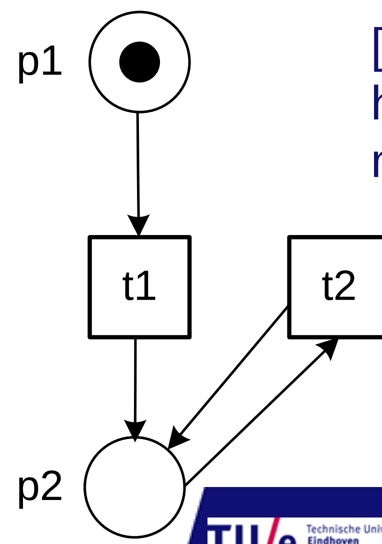
(N, M) is *reversible* if and only if for any $M' \in R(N, M)$: $M \in R(N, M')$.

Marking M' is a *home marking* in (N, M) if it is reachable from any reachable marking, i.e., for any $M'' \in R(N, M)$: $M' \in R(N, M'')$.

(N, M) is *reversible* if and only if M is a home marking.







[p2] is home marking

Reachability Graph

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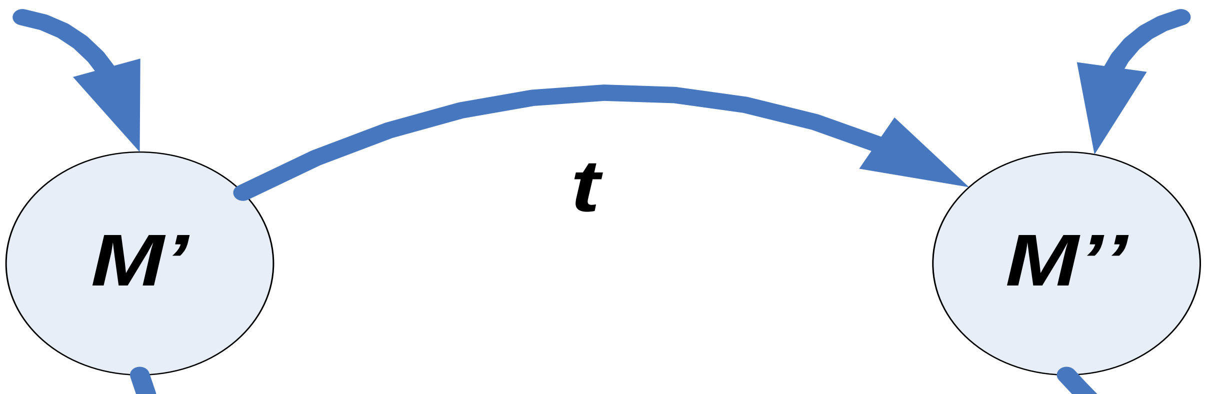
Definition

Definition 10 (Reachability graph). *Let* $N = (P, T, F, W)$ *be a Petri net and* $M \in \mathbb{B}(P)$ *be a marking. The reachability graph of* (N, M) *is the graph* (V, E) *with as vertices* $V = R(N, M)$ *the set of all reachable markings and as edges* $E = \{(M', t, M'') \in V \times T \times V \mid ((N, M')[t\rangle(N, M'')\}$ *the set of all possible state changes. Note that* $(M', t, M'') \in E$ *denotes that* M'' *is reachable from* M' *by firing* t.

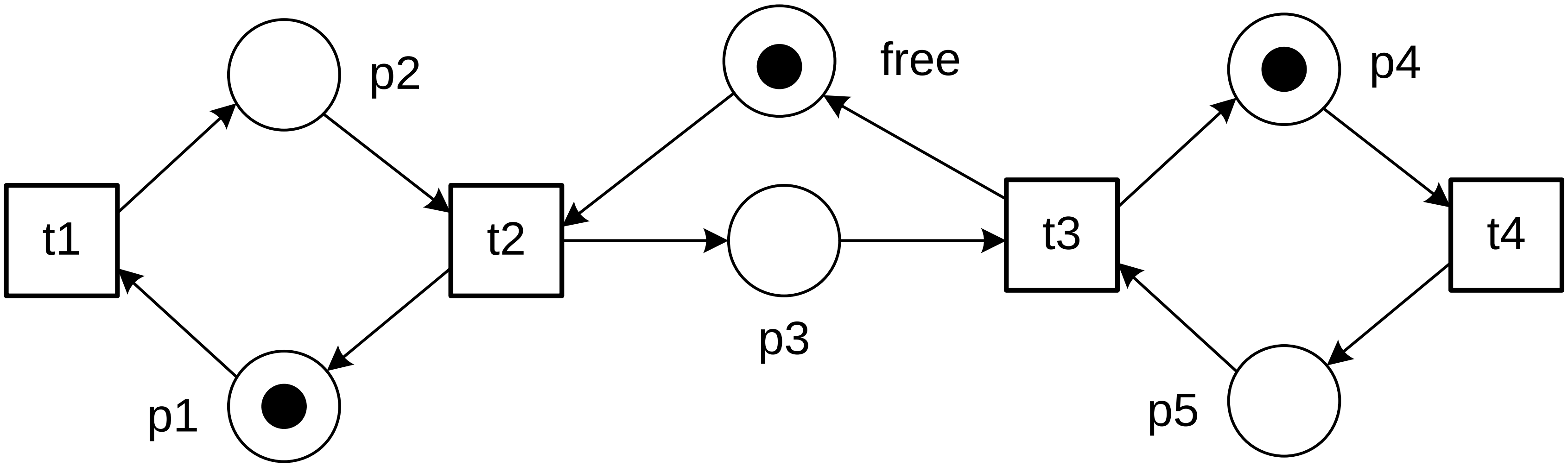


$$(N, M')[t\rangle(N, M'')$$

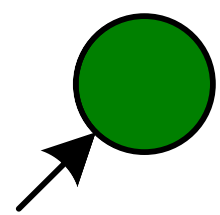
Reachability graph algorithm

1) Label the initial marking M_0 as the *root* and tag it "new".
2) While "new" markings exists, do the following:
   a) Select a new marking *M*.
   b) If no transitions are enabled at *M*, tag *M* "dead-end".
   c) While there exist enabled transitions at *M*, do the following for each enabled transition *t* at *M*:
      i. Obtain the marking *M'* that results from firing *t* at *M*.
      ii. If *M'* does not appear in the graph, add *M'* and tag it "new".
      iii. Draw an arc with label *t* from *M* to *M'* (if not already present).
3) Output the graph.

Example

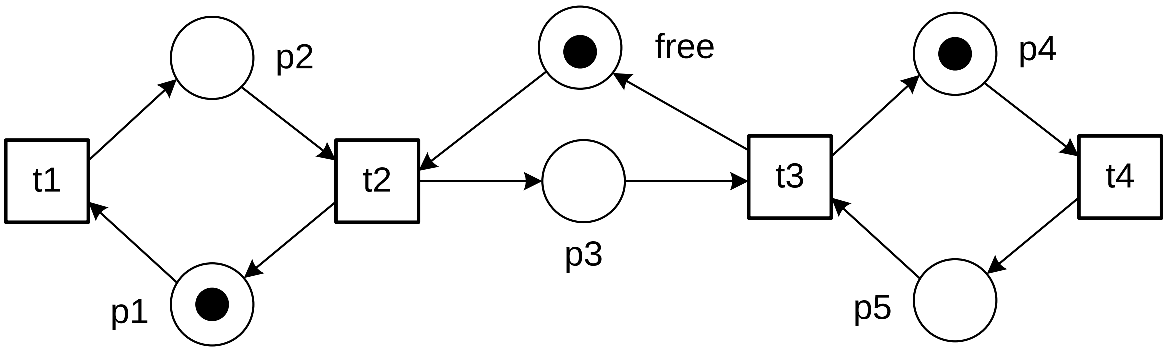


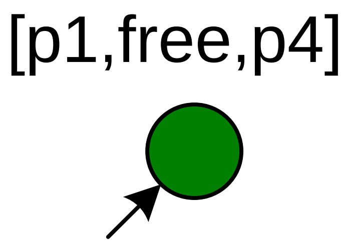
[p1,free,p4]

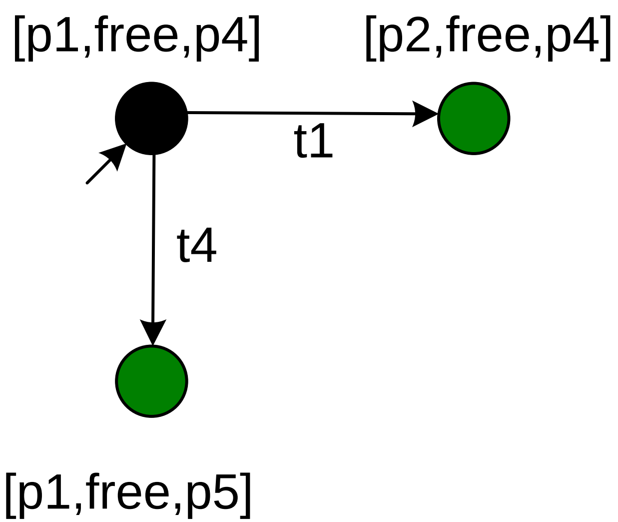


Step 1: Label the initial marking *M0* as the *root* and tag it "new" (indicated by green color).

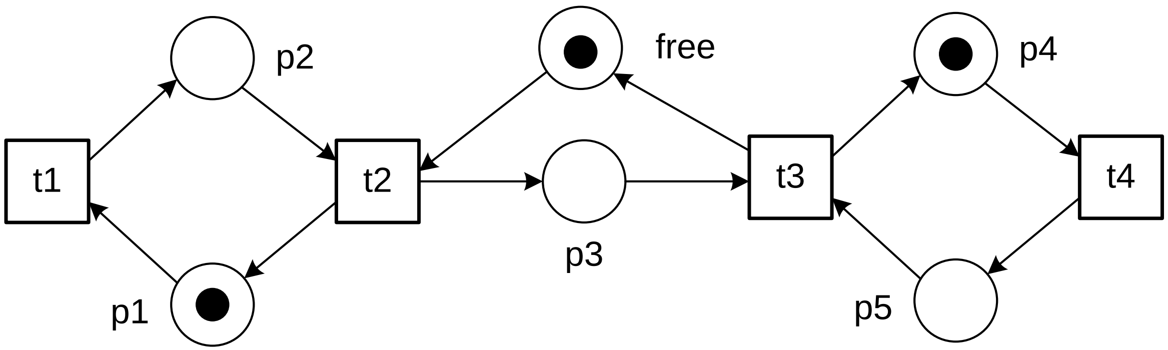
Example (continued)

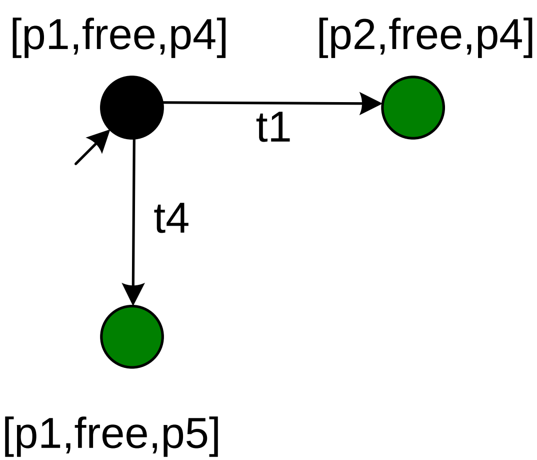


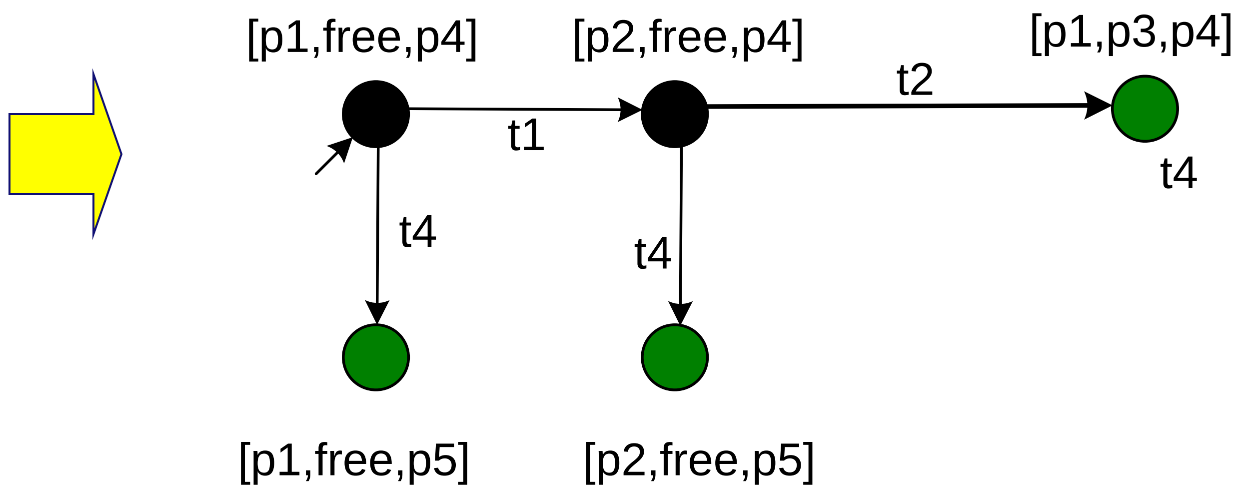




Example (continued)







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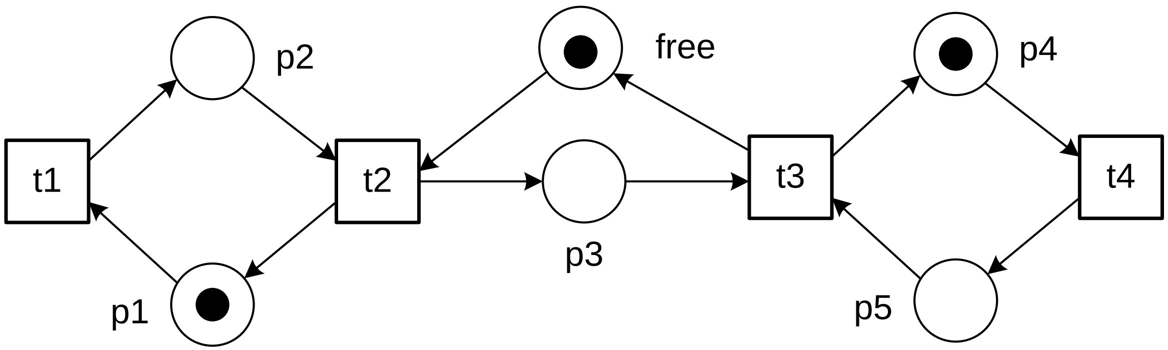
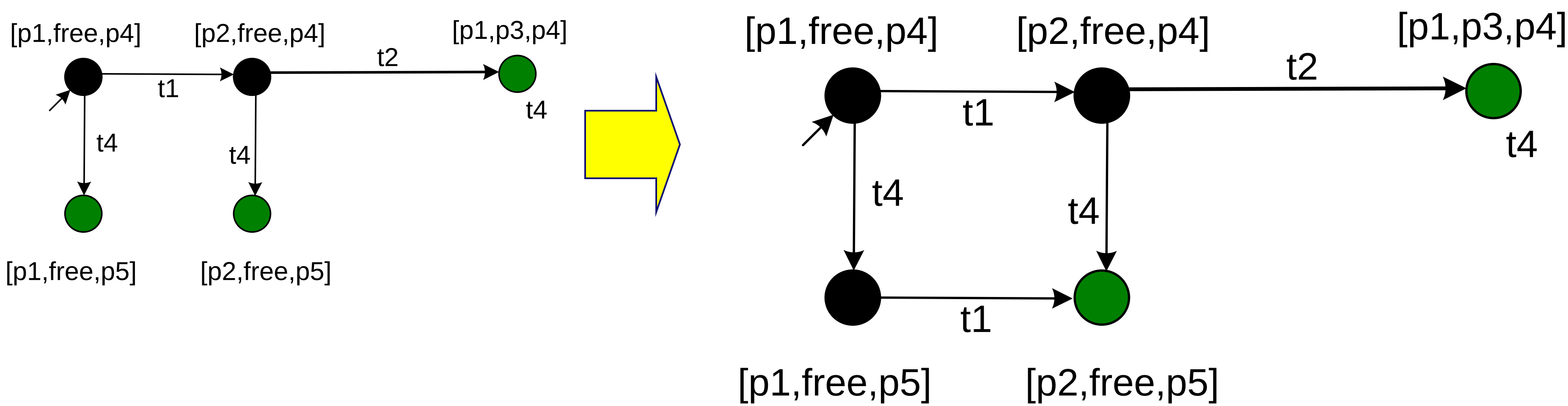
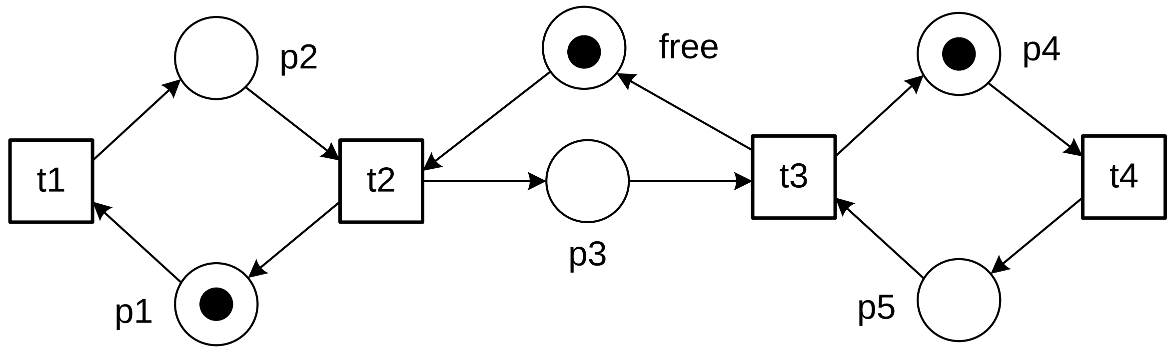
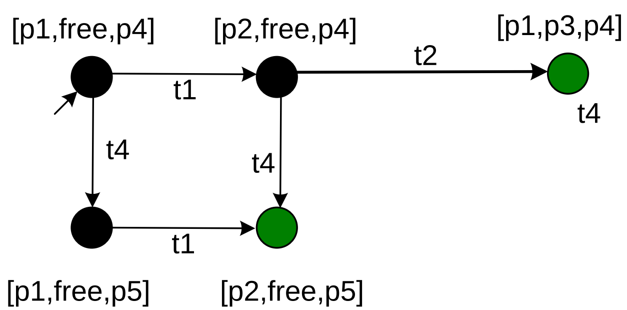
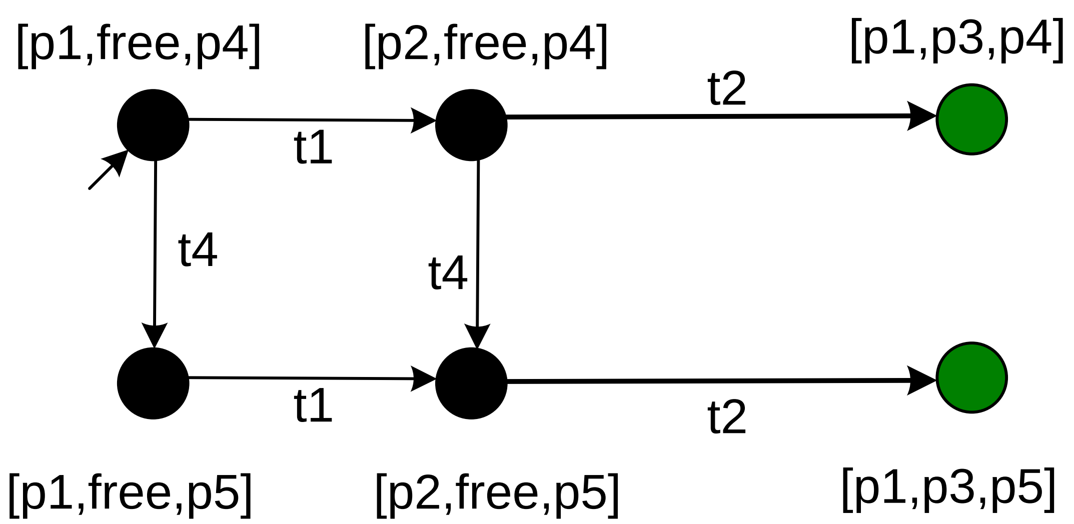




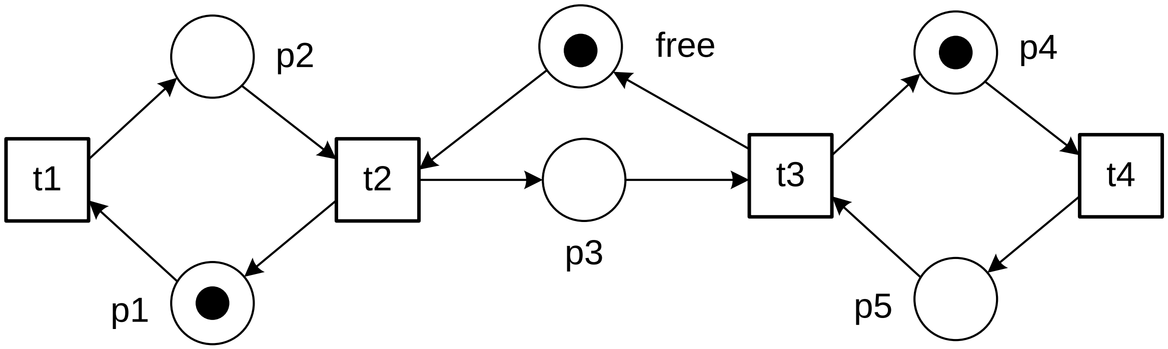
Example (continued)

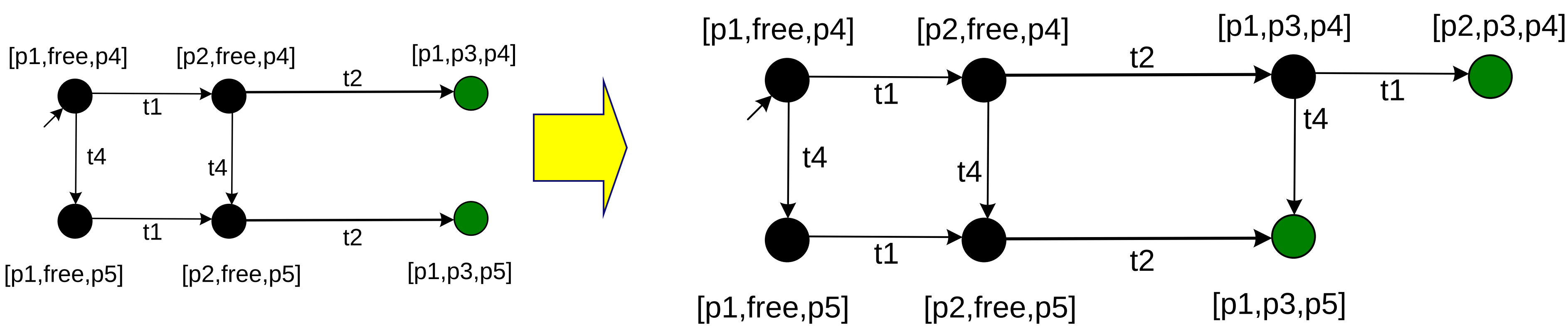






Example (continued)





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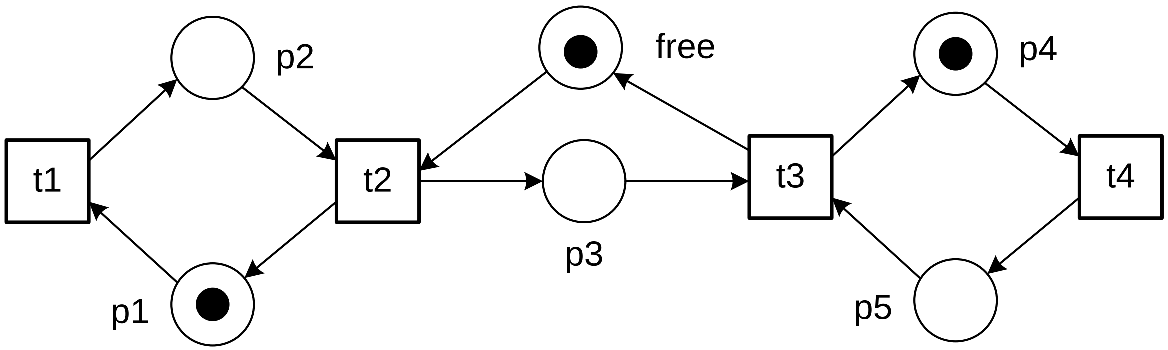
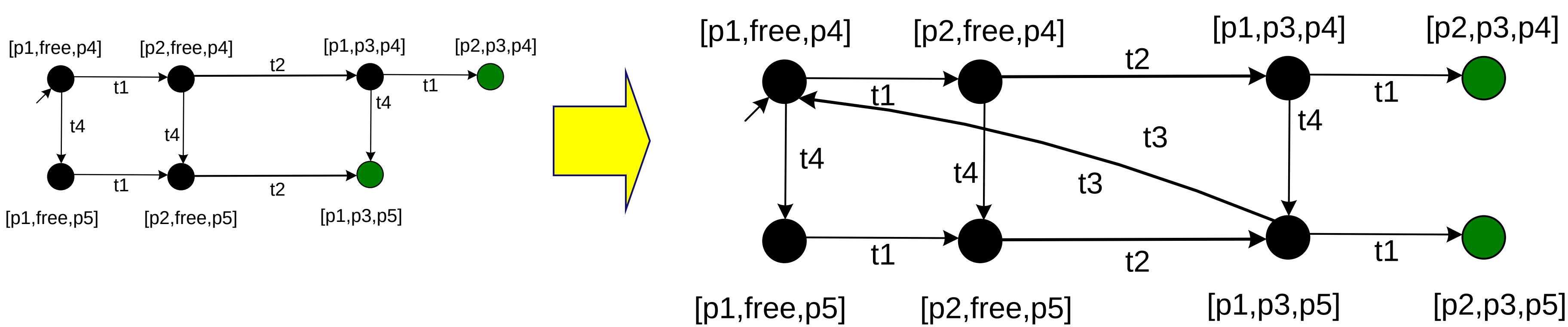
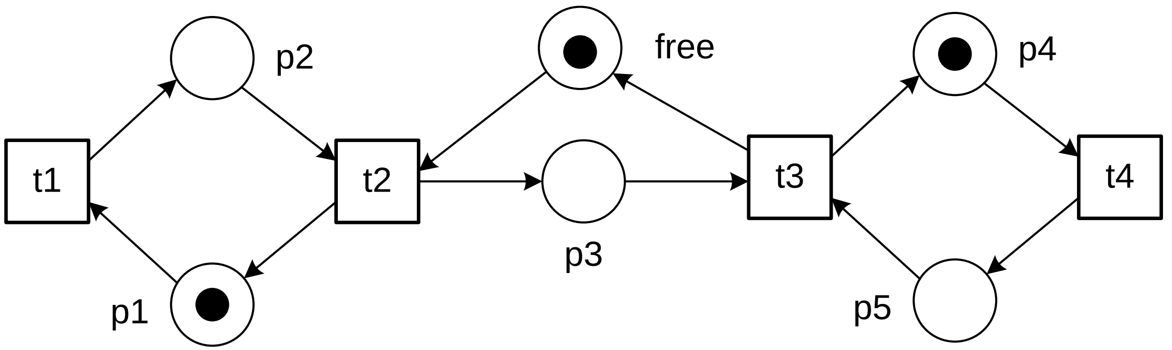
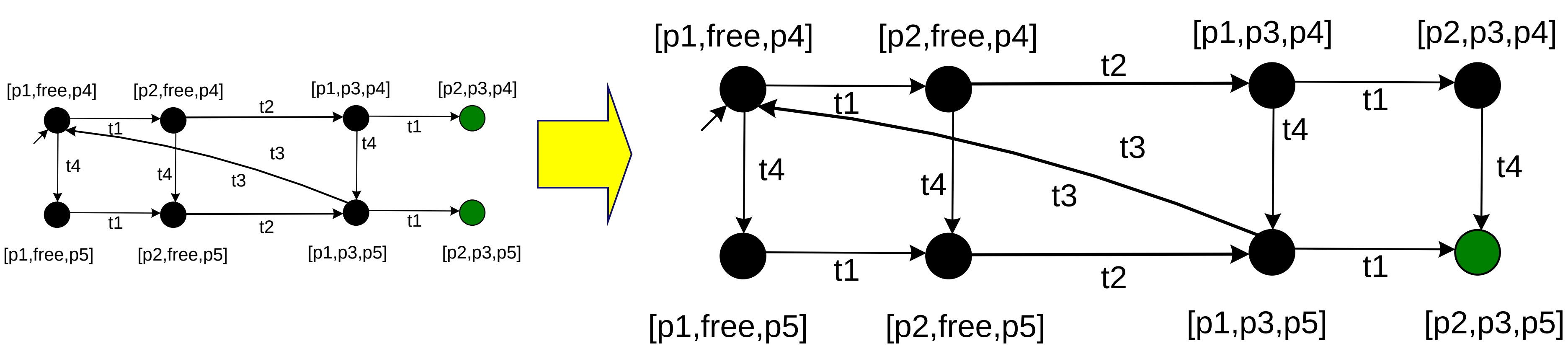




Example (continued)





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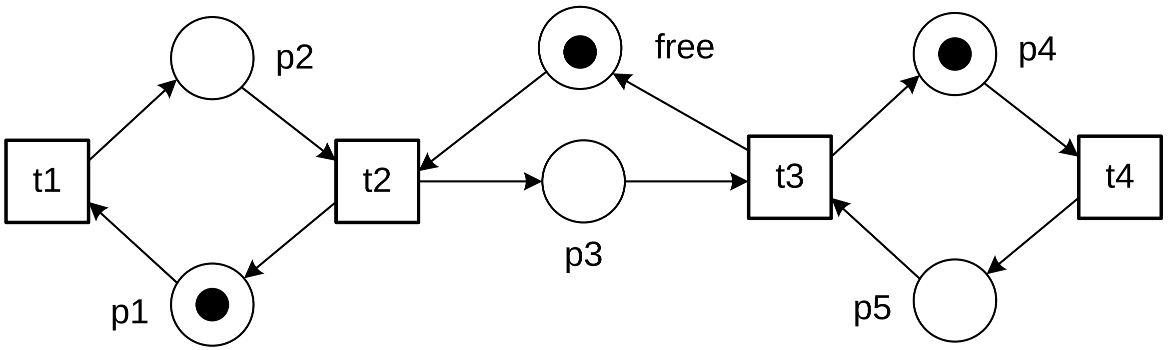
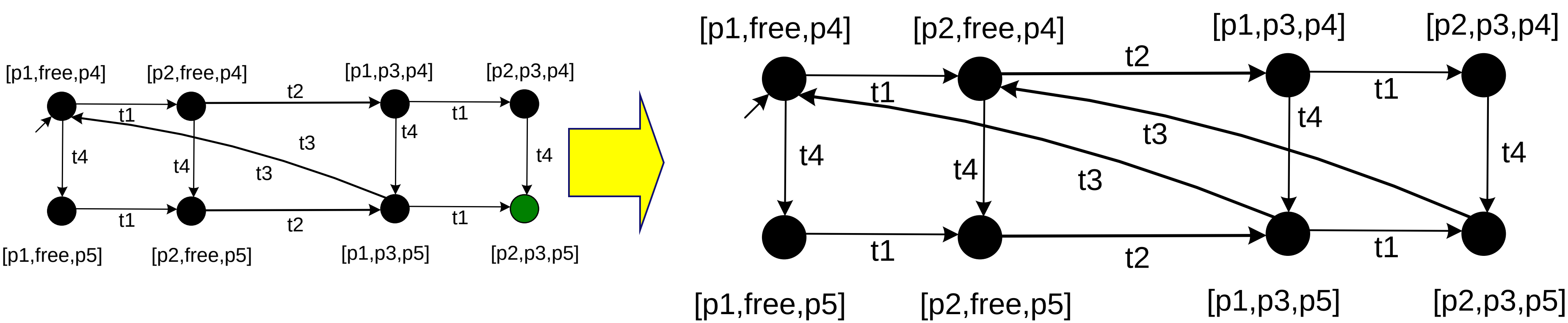
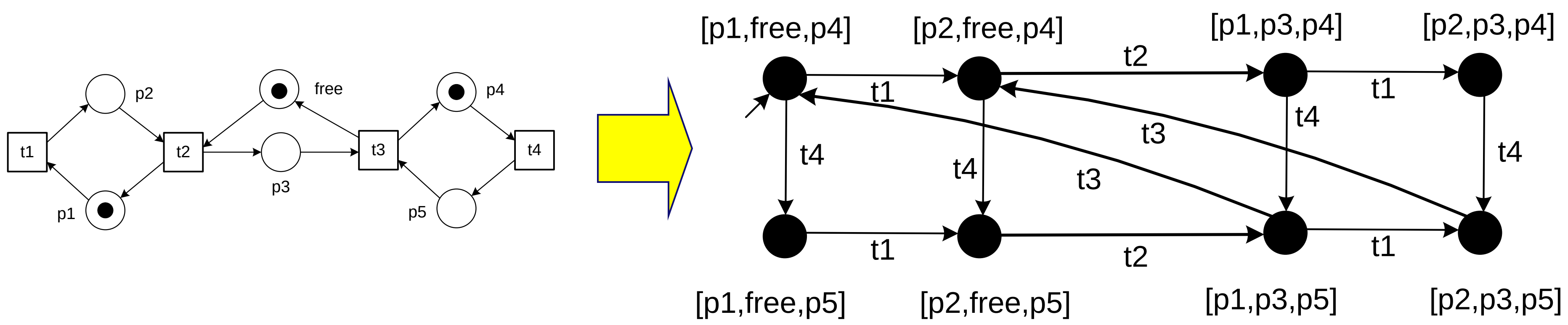




Example (complete)



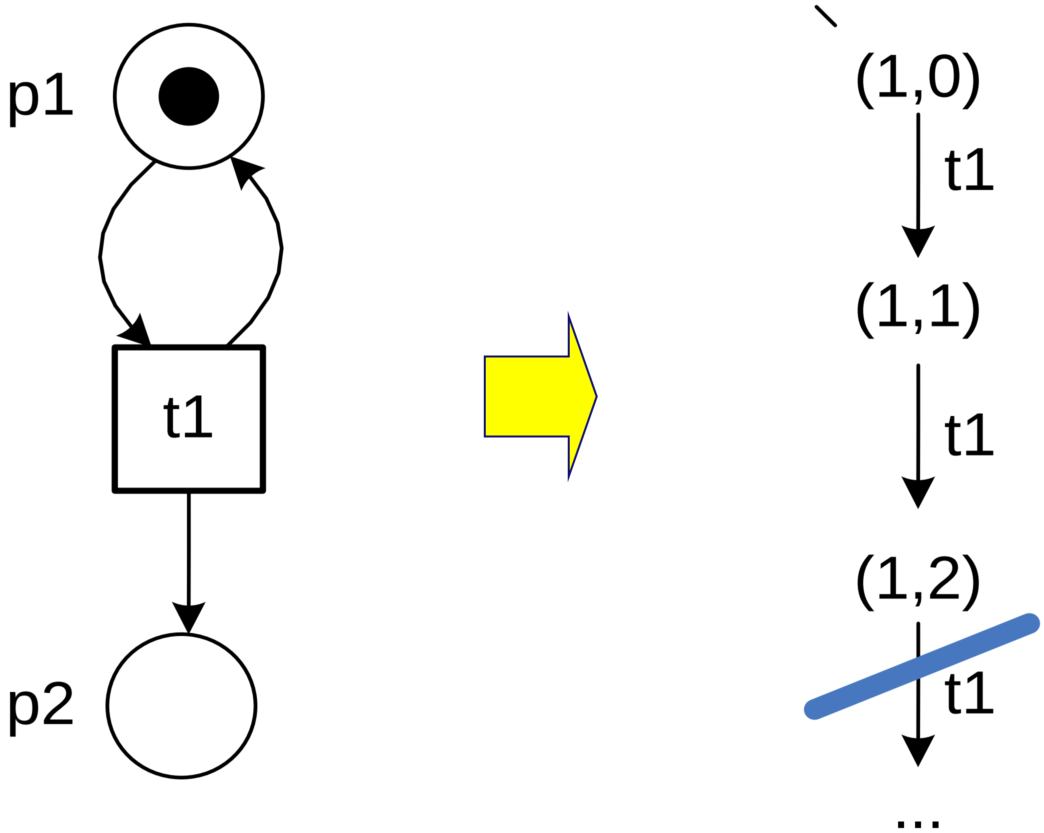
- **The marked Petri net is:**
  - ✓ **deadlock free**
  - ✓ **live**
  - ✓ **bounded**
  - ✓ **safe**
  - ✓ **reversible**
  - ✓ **all markings are home markings**

Coverability Graph

Technische Universiteit
Eindhoven
University of Technology

Where innovation starts

Problem

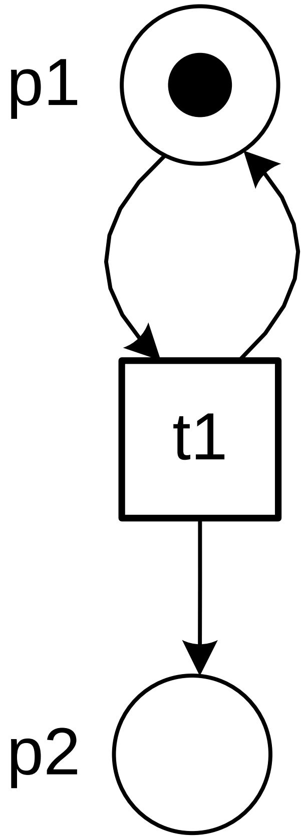


ps. (n,m) is a shorthand for [p1^n,p2^m]

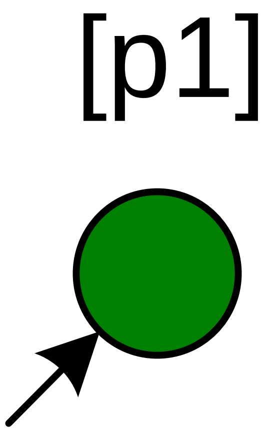
Coverability tree algorithm

1) Label the initial marking M_0 as the *root* and tag it "new".
2) While "new" markings exists, do the following:
   a) Select a new marking M and remove the "new" tag.
   b) If M is identical to a marking on the path from the *root* to M, then tag M "old" and go to another new marking.
   c) If no transitions are enabled at M, tag M "dead-end".
   d) While there exist enabled transitions at M, do the following for each enabled transition t at M:
      i. Obtain the marking M' that results from firing t at M.
      ii. If, on the path from the *root* to M, there exists a marking M'' such that $M'(p) \geq M''(p)$ for each p and $M' \neq M''$ (i.e., M'' is coverable), then replace $M'(p)$ by ω for each p such that $M'(p) > M''(p)$.
      iii. Introduce M' as a node, draw an arc with label t from M to M', and tag M' "new".
3) Output the tree.

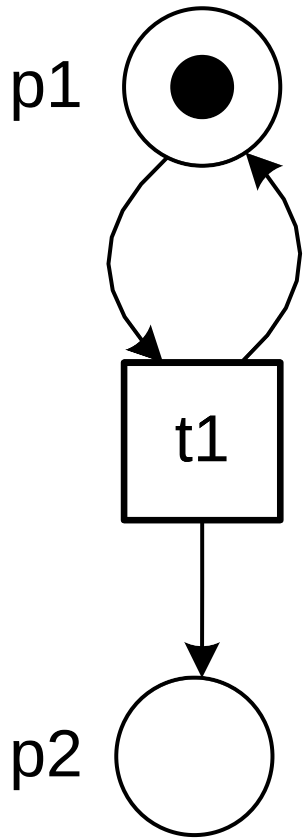
Example



Step 1: Label the initial marking M_0 as the *root* and tag it "new" (indicated by green color).

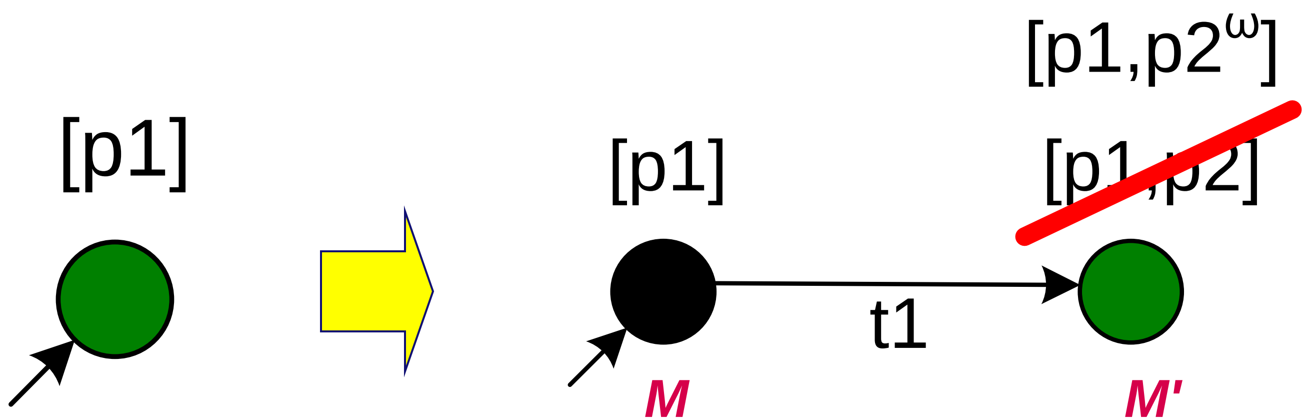


Example (continued)



Step 2: While "new" markings exists, do the following:

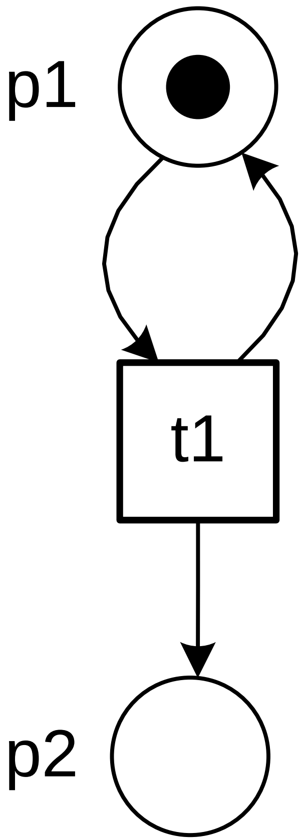
- **Select a new marking *M* and remove the "new" tag.**
- If *M* is identical to a marking on the path from the *root* to *M*, then tag *M* "old" and go to another new marking.
- If no transitions are enabled at *M*, tag *M* "dead-end".
- **While there exist enabled transitions at *M*, do the following for each enabled transition *t* at *M*:**
  - **Obtain the marking *M'* that results from firing *t* at *M*.**
  - **If, on the path from the *root* to *M*, there exists a marking *M''* such that *M'(p)* ≥ *M''(p)* for each *p* and *M'*≠*M''* (i.e., *M''* is coverable), then replace *M'(p)* by *ω* for each *p* such that *M'(p)* > *M''(p)*.**
  - **Introduce *M'* as a node, draw an arc with label *t* from *M* to *M'*, and tag *M'* "new"**

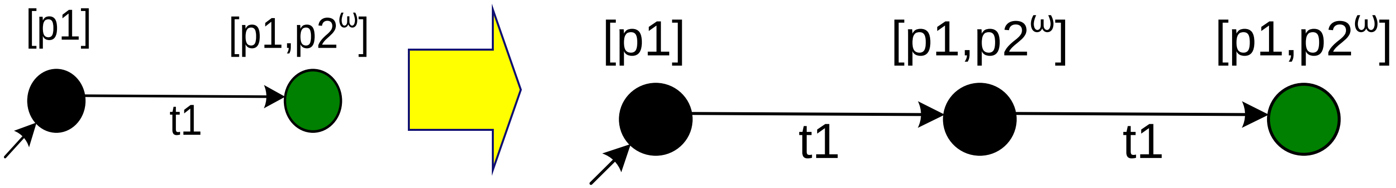


Example (continued)

Step 2: While "new" markings exists, do the following:

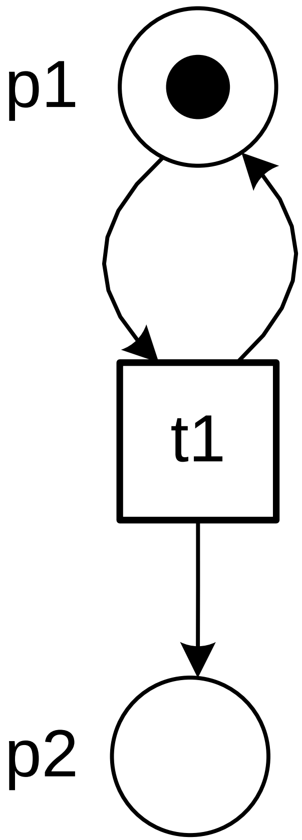
- **Select a new marking *M* and remove the "new" tag.**
- If *M* is identical to a marking on the path from the *root* to *M*, then tag *M* "old" and go to another new marking.
- If no transitions are enabled at *M*, tag *M* "dead-end".
- **While there exist enabled transitions at *M*, do the following for each enabled transition *t* at *M*:**
  - **Obtain the marking *M'* that results from firing *t* at *M*.**
  - **If, on the path from the *root* to *M*, there exists a marking *M''* such that *M'(p)* ≥ *M''(p)* for each *p* and *M'≠M''* (i.e., *M''* is coverable), then replace *M'(p)* by *ω* for each *p* such that *M'(p)* > *M''(p)*.**
  - **Introduce *M'* as a node, draw an arc with label *t* from *M* to *M'*, and tag *M'* "new"**





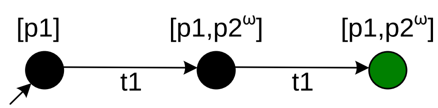
$\omega+k = \omega-k = \omega$

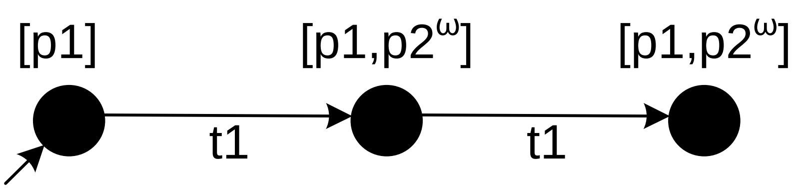
Example (continued)



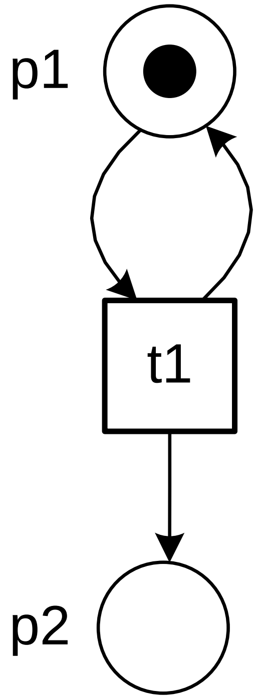
Step 2: While "new" markings exists, do the following:

- **Select a new marking *M* and remove the "new" tag.**
- **If *M* is identical to a marking on the path from the *root* to *M*, then tag *M* "old" and go to another new marking.**
- If no transitions are enabled at *M*, tag *M* "dead-end".
- While there exist enabled transitions at *M*, do the following for each enabled transition *t* at *M*:
  - Obtain the marking *M'* that results from firing *t* at *M*.
  - If, on the path from the *root* to *M*, there exists a marking *M''* such that $M'(p) \geq M''(p)$ for each *p* and $M' \neq M''$ (i.e., *M''* is coverable), then replace $M'(p)$ by ω for each *p* such that $M'(p) > M''(p)$.
  - Introduce *M'* as a node, draw an arc with label *t* from *M* to *M'*, and tag *M'* "new"

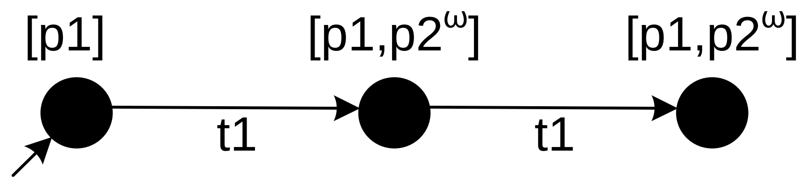




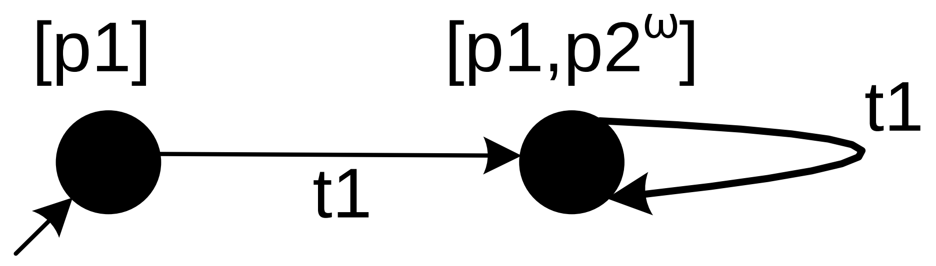
Example (complete)



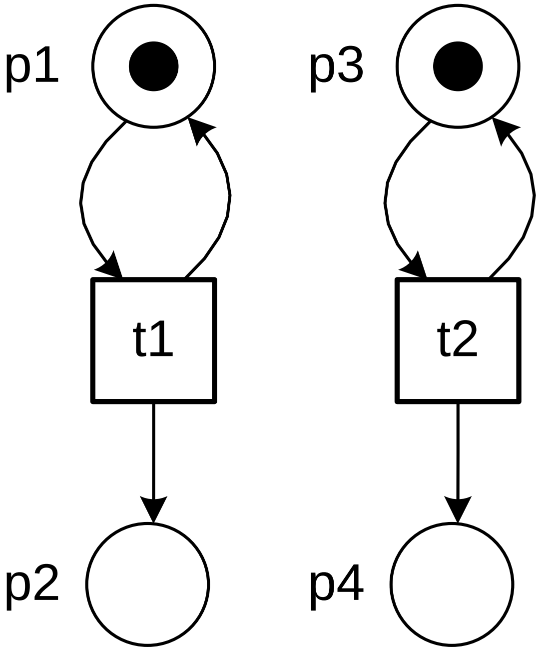
Step 3: Output the tree.

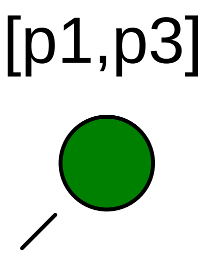


Coverability graph:

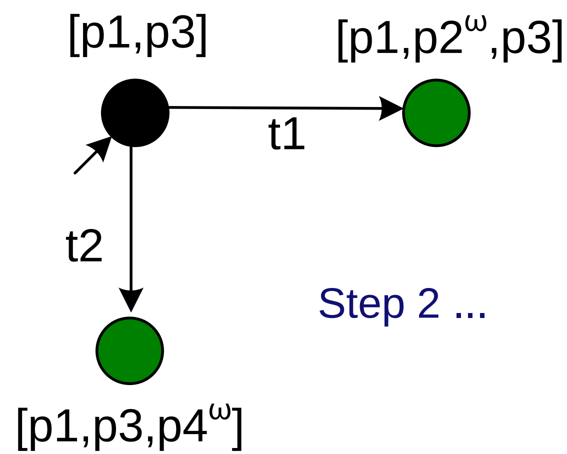


Another example



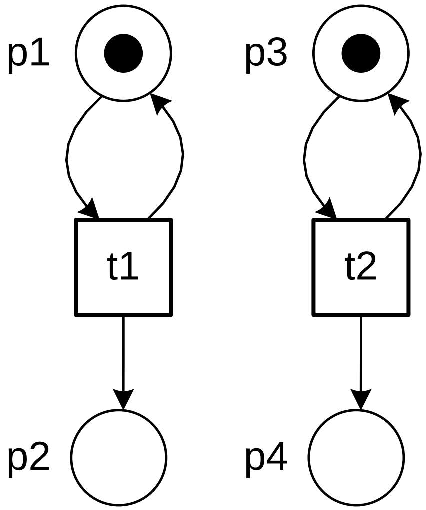


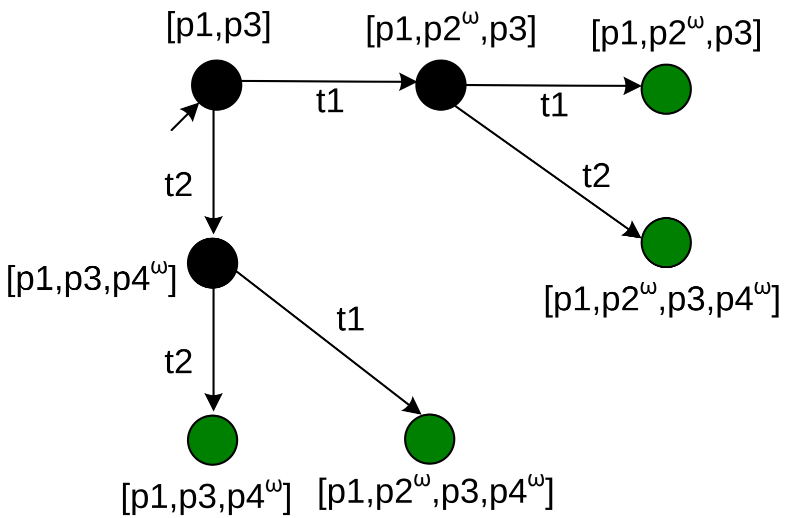
Step 1: Label the initial marking M_0 as the *root* and tag it "new" (indicated by green color).

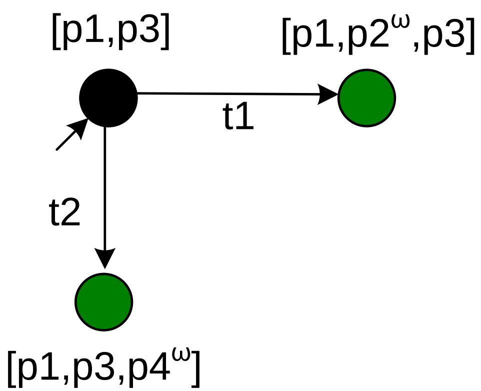


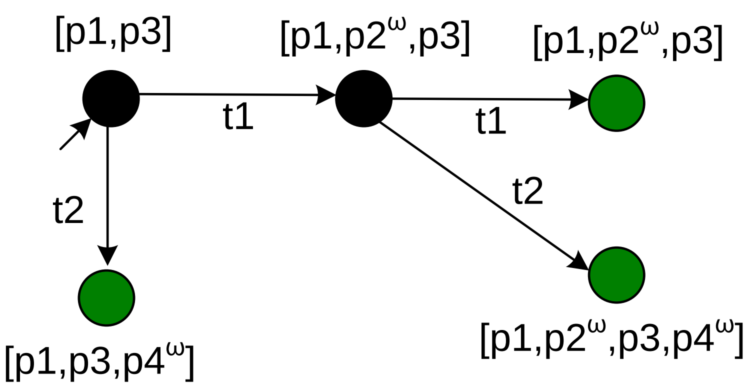
Step 2 ...

Example (continued)

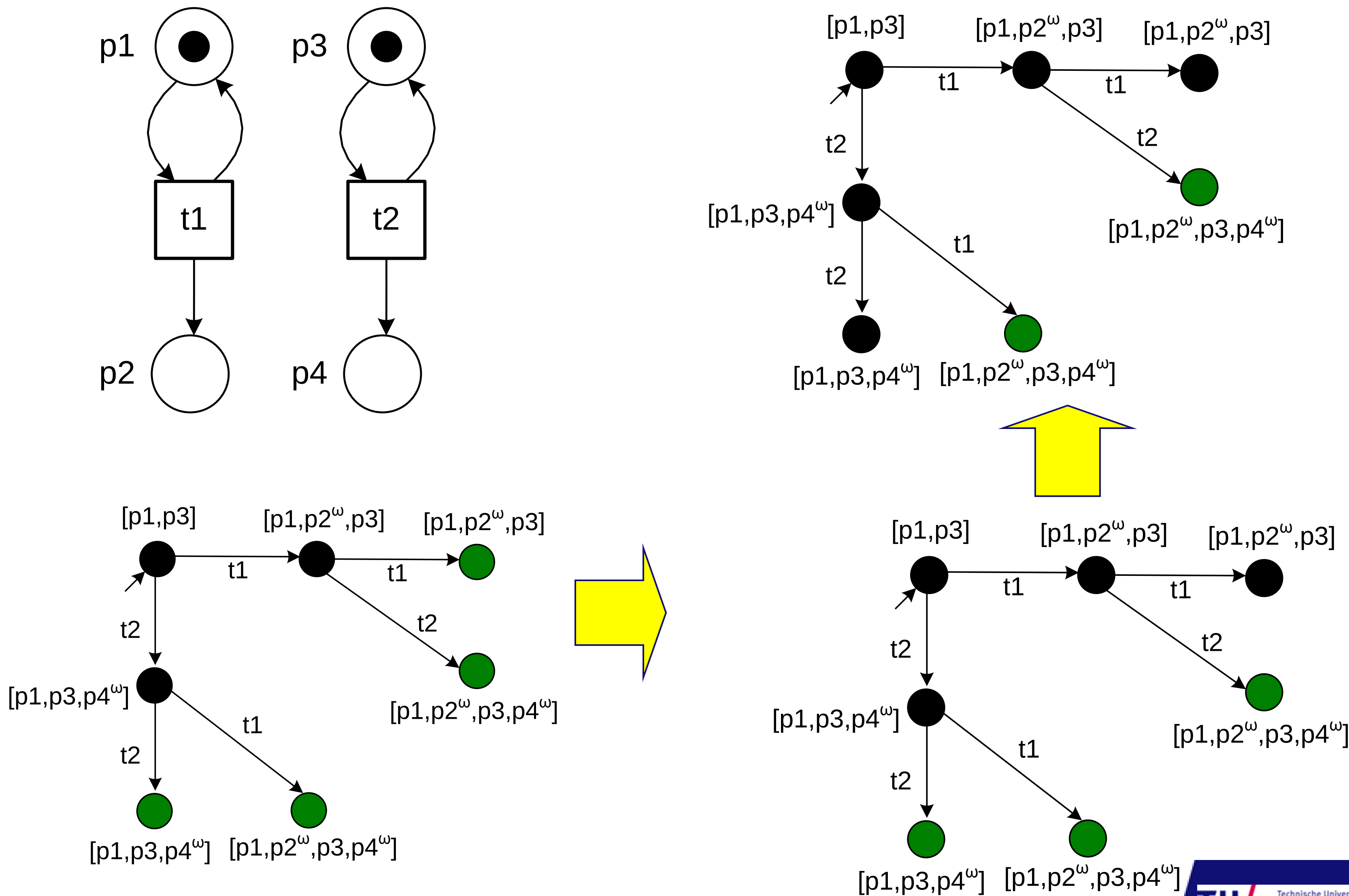




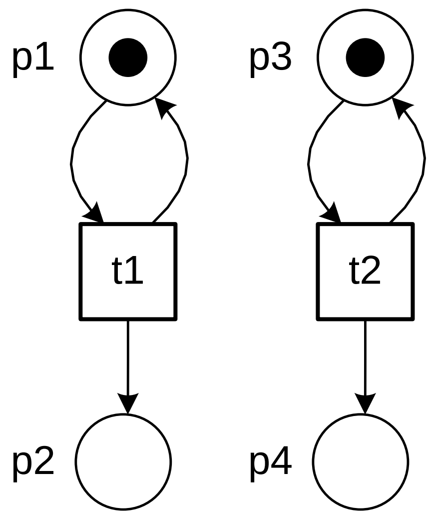


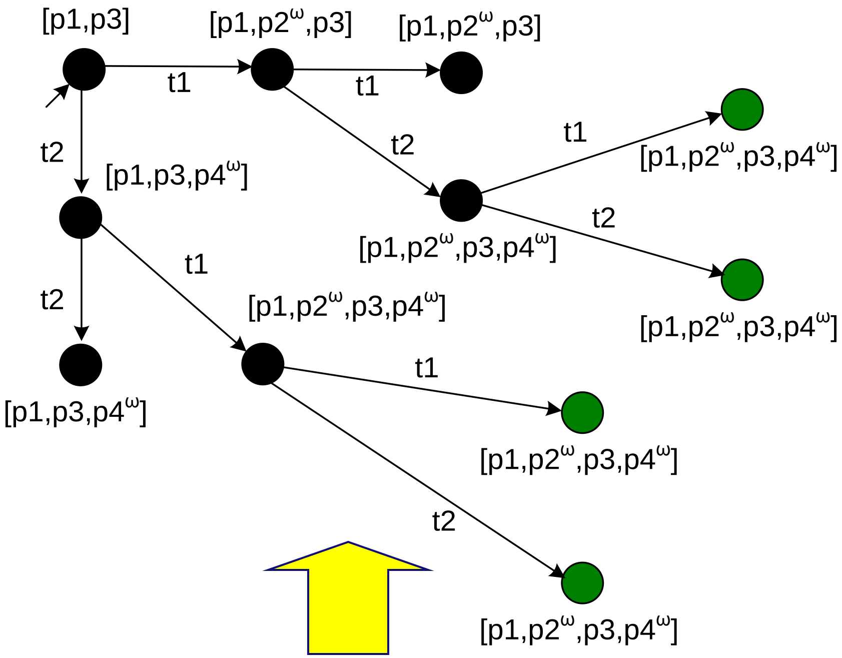


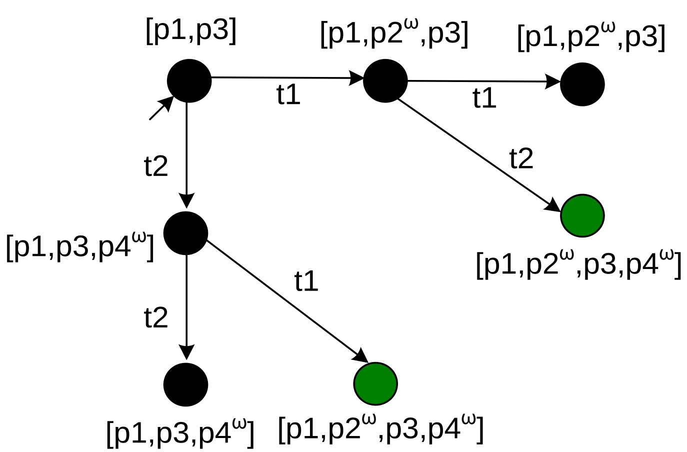
Example (continued)

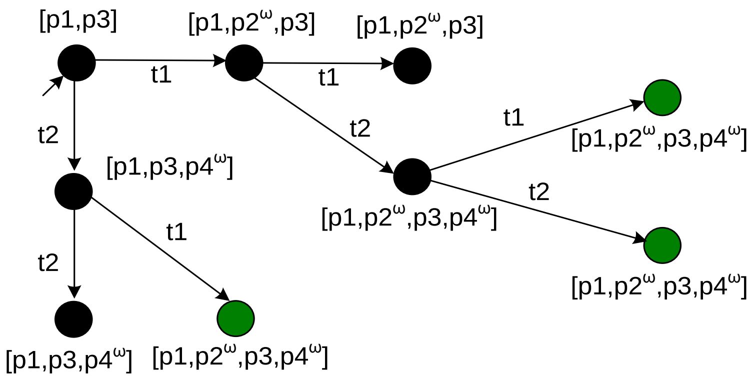


Example (continued)

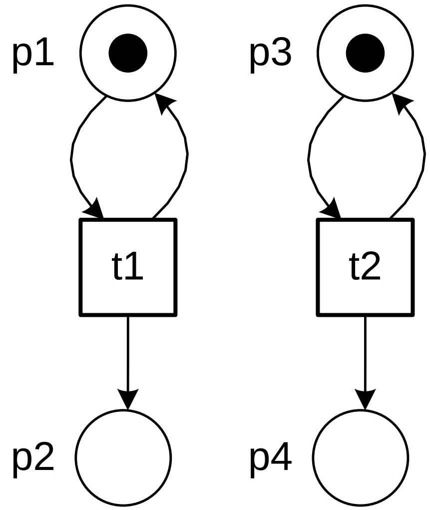




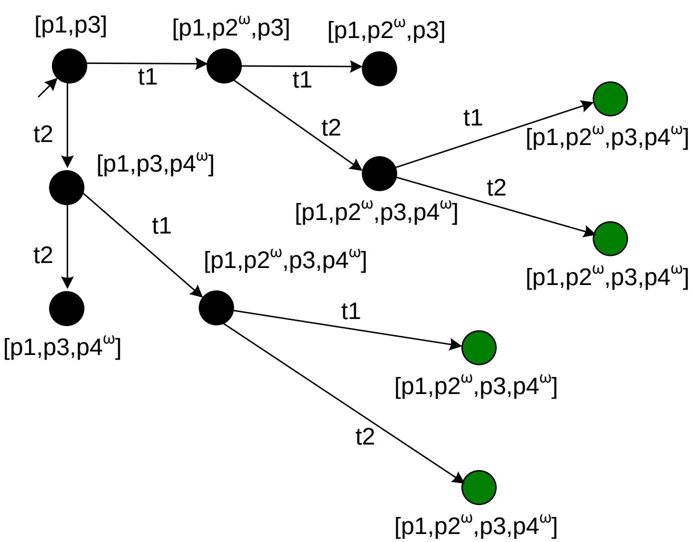


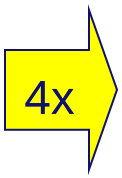


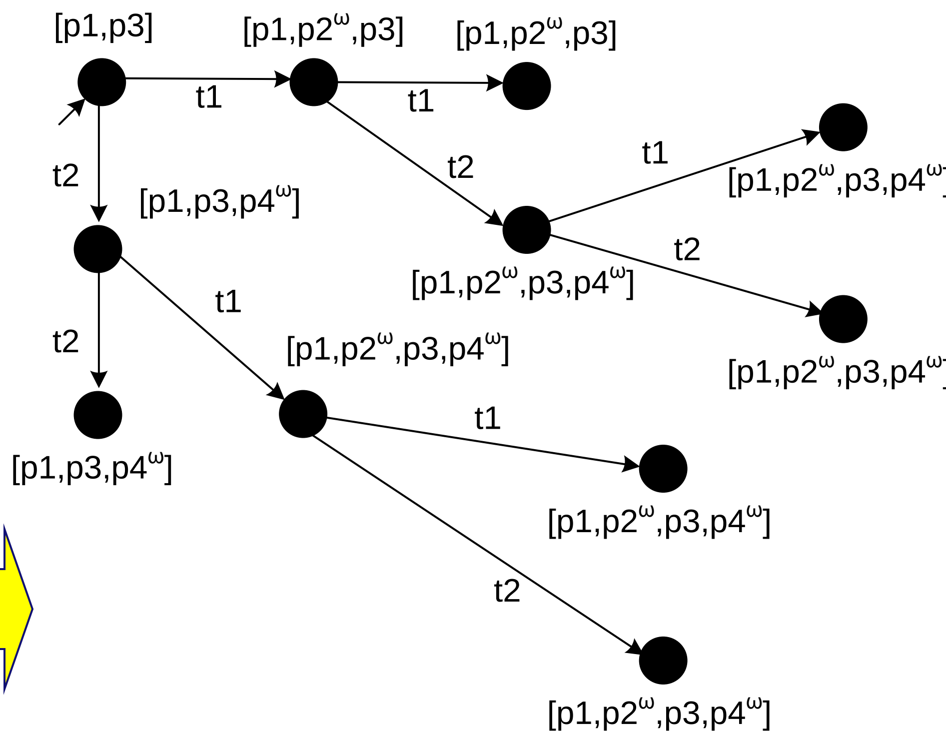
Example (continued)



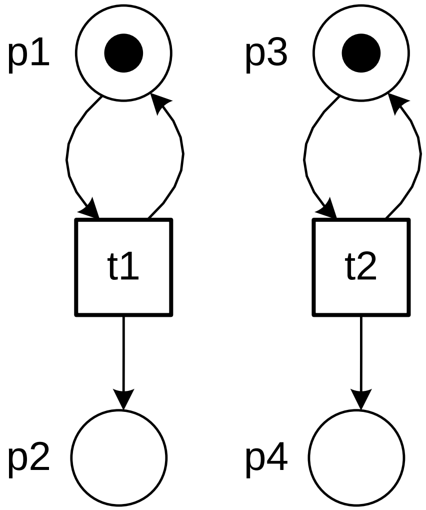
Step 3: Output the tree.

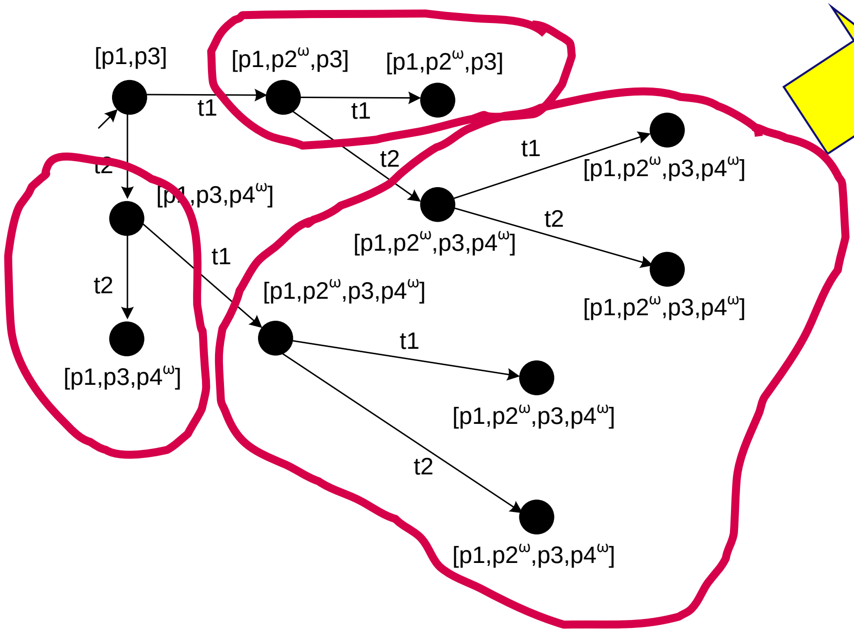


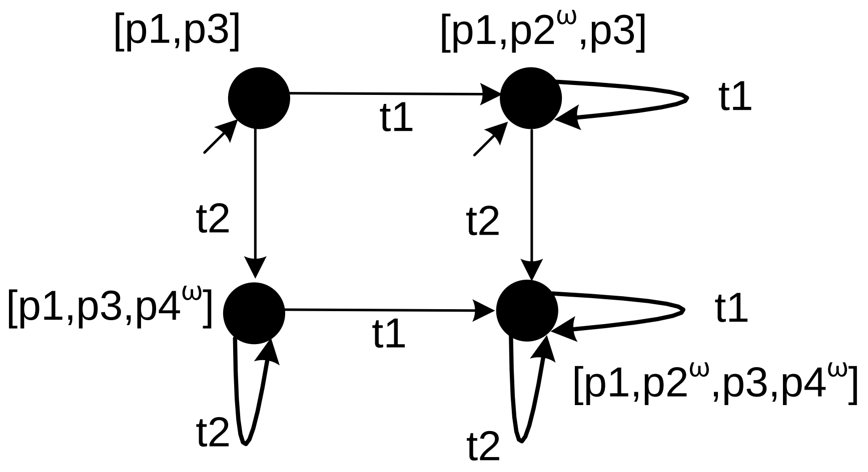




Example (complete)



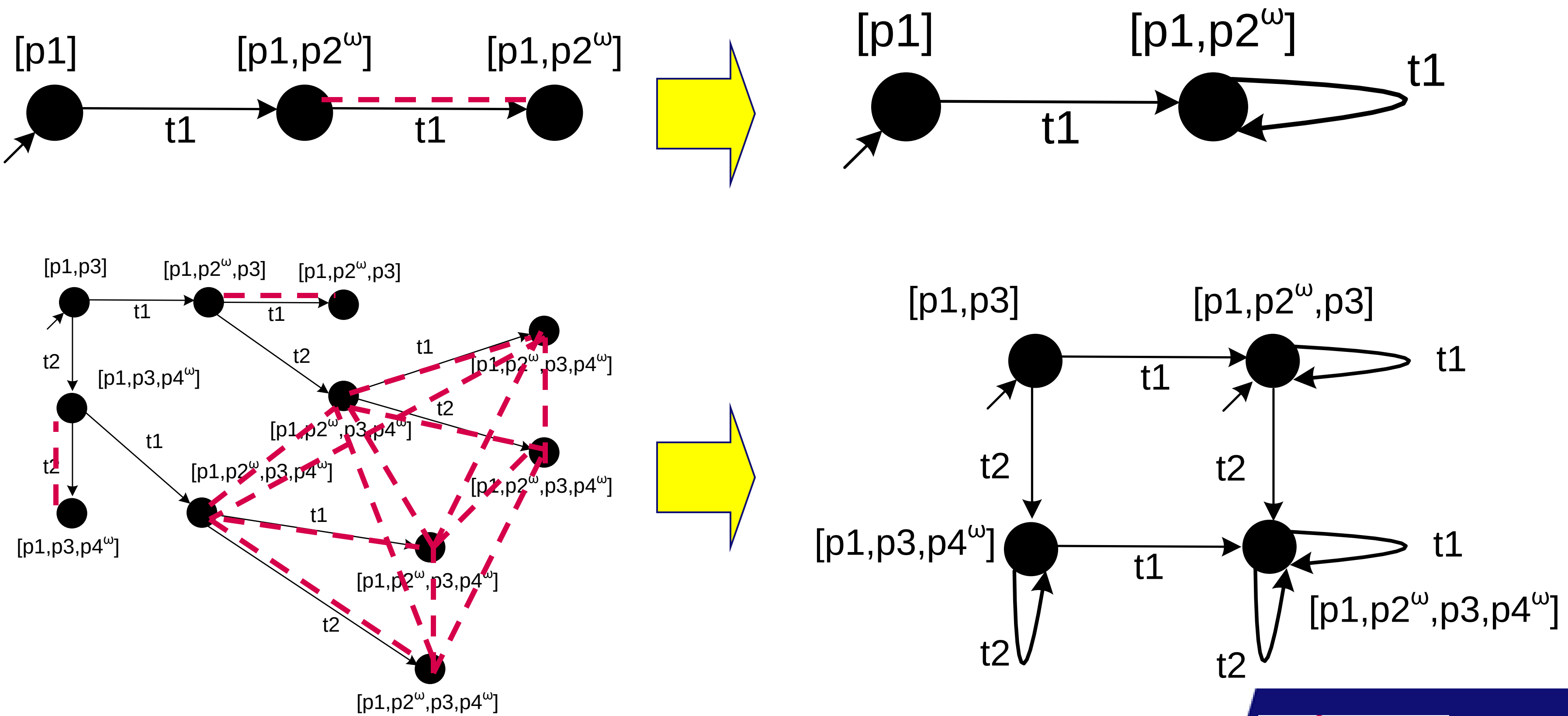




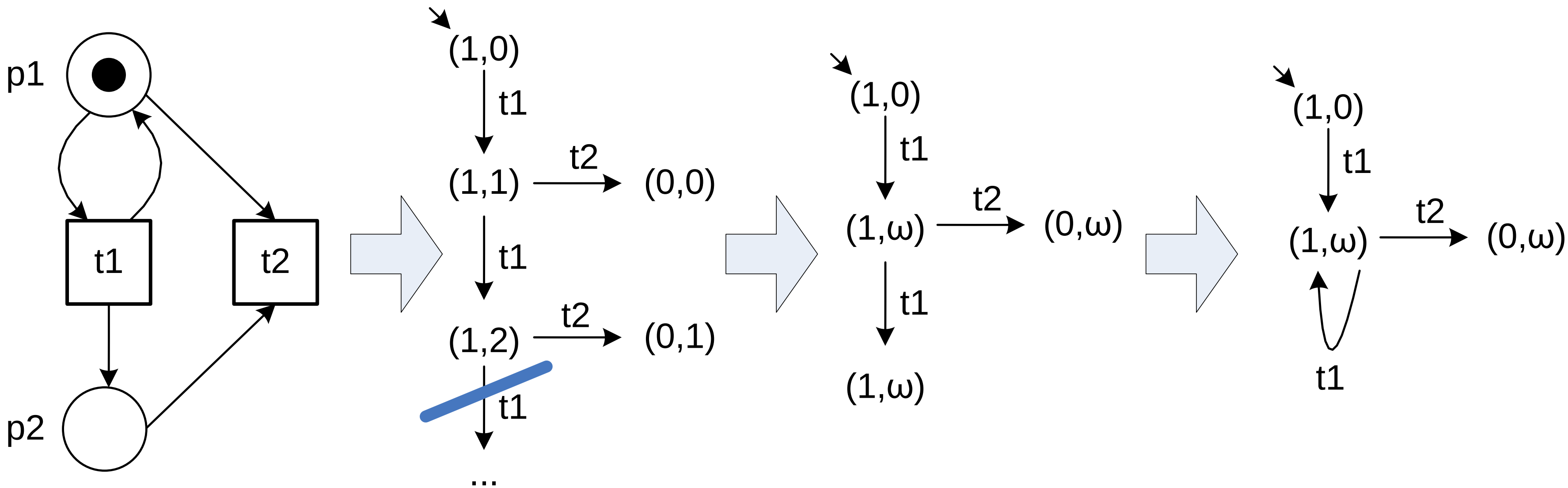
Coverability graph

Coverability graph

- **Take the coverability tree and simply merge nodes with identical labels**



Another example



marked net — *reachability graph* — *coverability tree* — *coverability graph*

ps. (n,m) is a shorthand for [p1^n,p2^m]

ω-markings

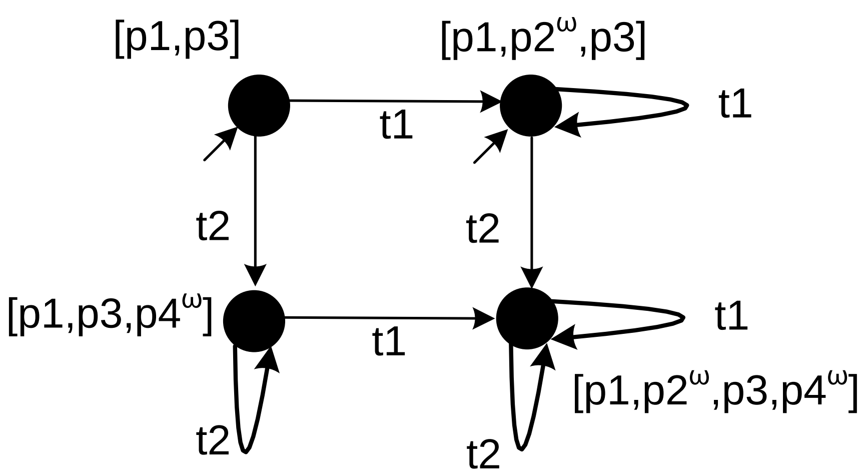
Definition 11 (ω-marking). *Let* $N = (P, T, F, W)$ *be a Petri net with initial marking* M'.

An ω-marking M *of* N *is an extended multi-set over* P, *i.e.,* $M \in A \rightarrow (\mathbb{N} \cup \{\omega\})$.

If $M(p) = \omega$, *then place* $p \in P$ *is said to be unbounded in* M.

If $M(p) \neq \omega$ *for all* $p \in P$, *then* M *is said to be ω-free.*

M *is a reachable ω-marking of* (N, M') *if and only if it appears in the coverability graph of* (N, M').



Properties

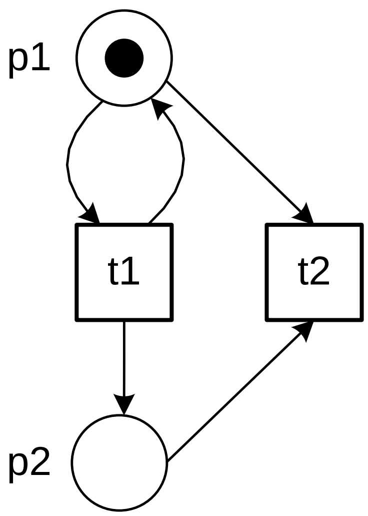
- The coverability tree/graph is always finite.
- The marked Petri net is bounded if and only if the corresponding coverability tree/graph contains only ω-free markings.
- The coverability tree/graph gives an over-approximation.
- Different Petri nets may have the same coverability tree/graph.

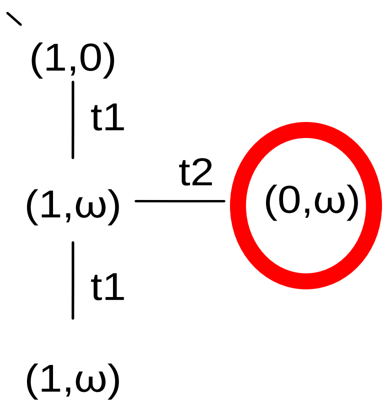
Basic relation between reachable markings and coverability tree/graph

Theorem 1 (Relation). *Let $N = (P, T, F, W)$ be a Petri net and $M \in \mathbb{B}(P)$ be a marking. Let M' be an ω-marking appearing in the coverability graph of (N, M) and $n \in \mathbb{N}$ an arbitrary number.*

There exists an $M'' \in R(N, M)$ such that for all $p \in P$:

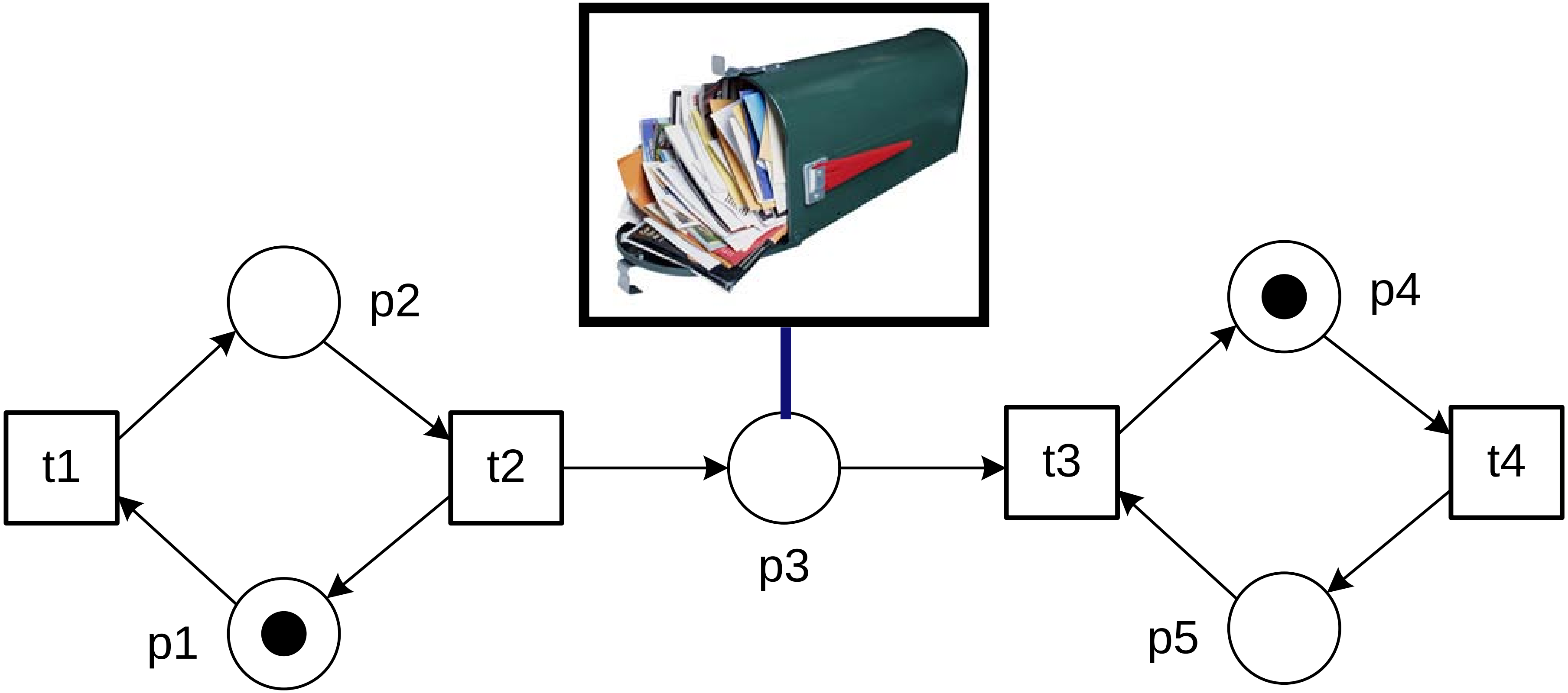
- *If $M'(p) \neq \omega$, then $M''(p) = M'(p)$.*
- *If $M'(p) = \omega$, then $M''(p) \geq n$.*





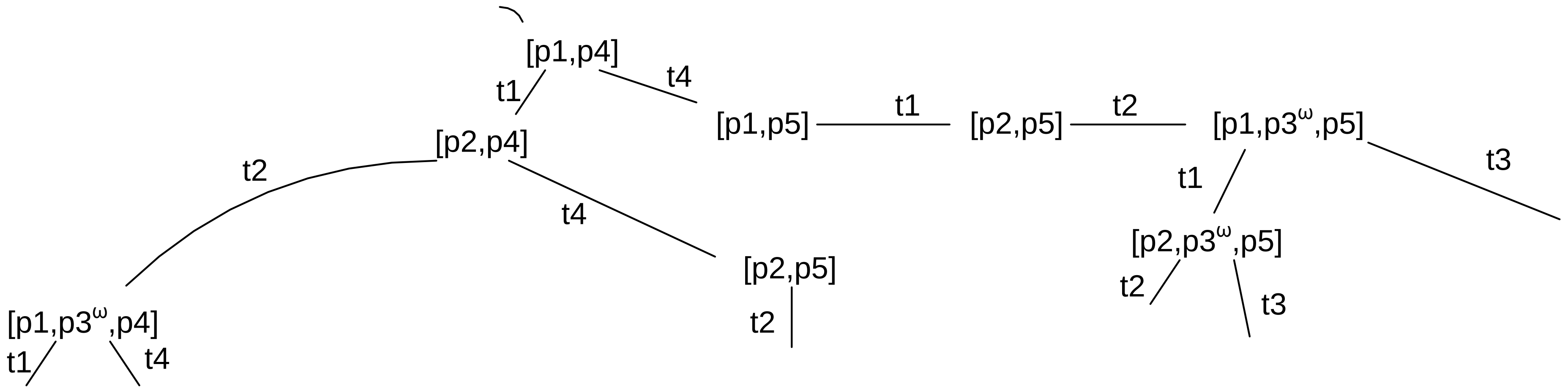
Let n=180. There is a reachable marking with 0 tokens in p1 and at least 180 tokens in p2.

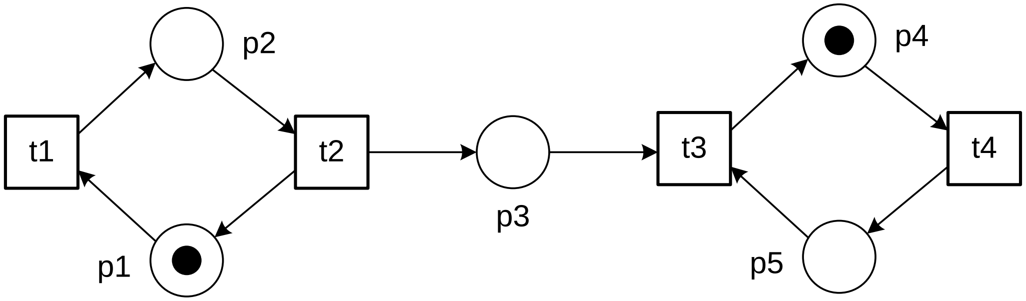
Example (readers and writers)



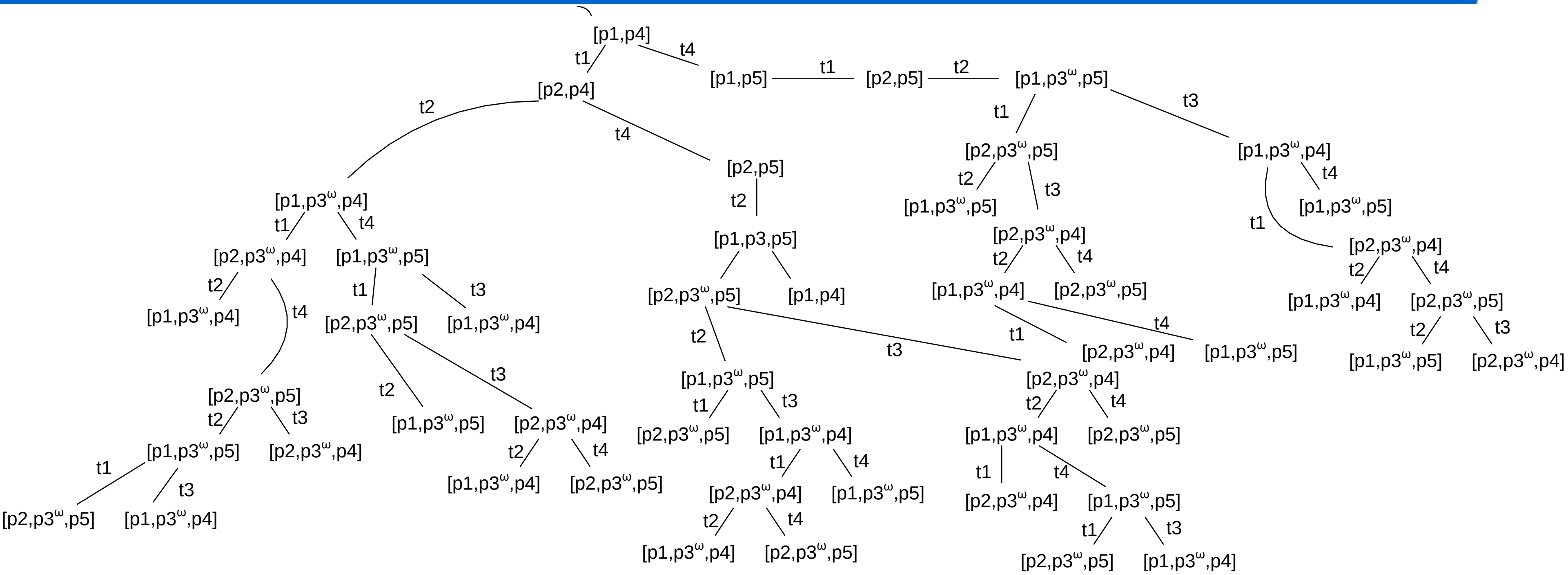
construct coverability graph ...

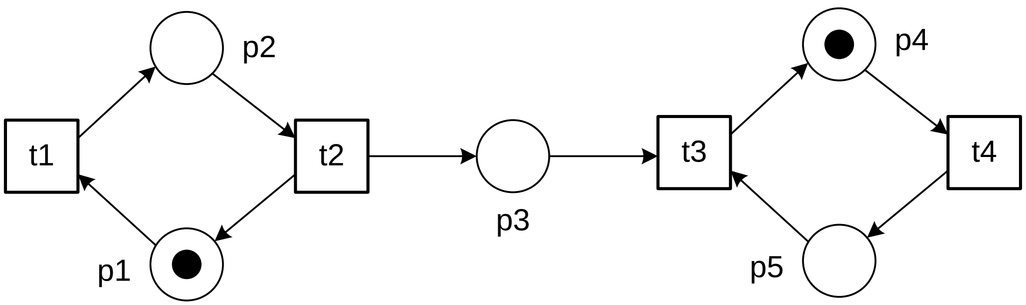
Initial part



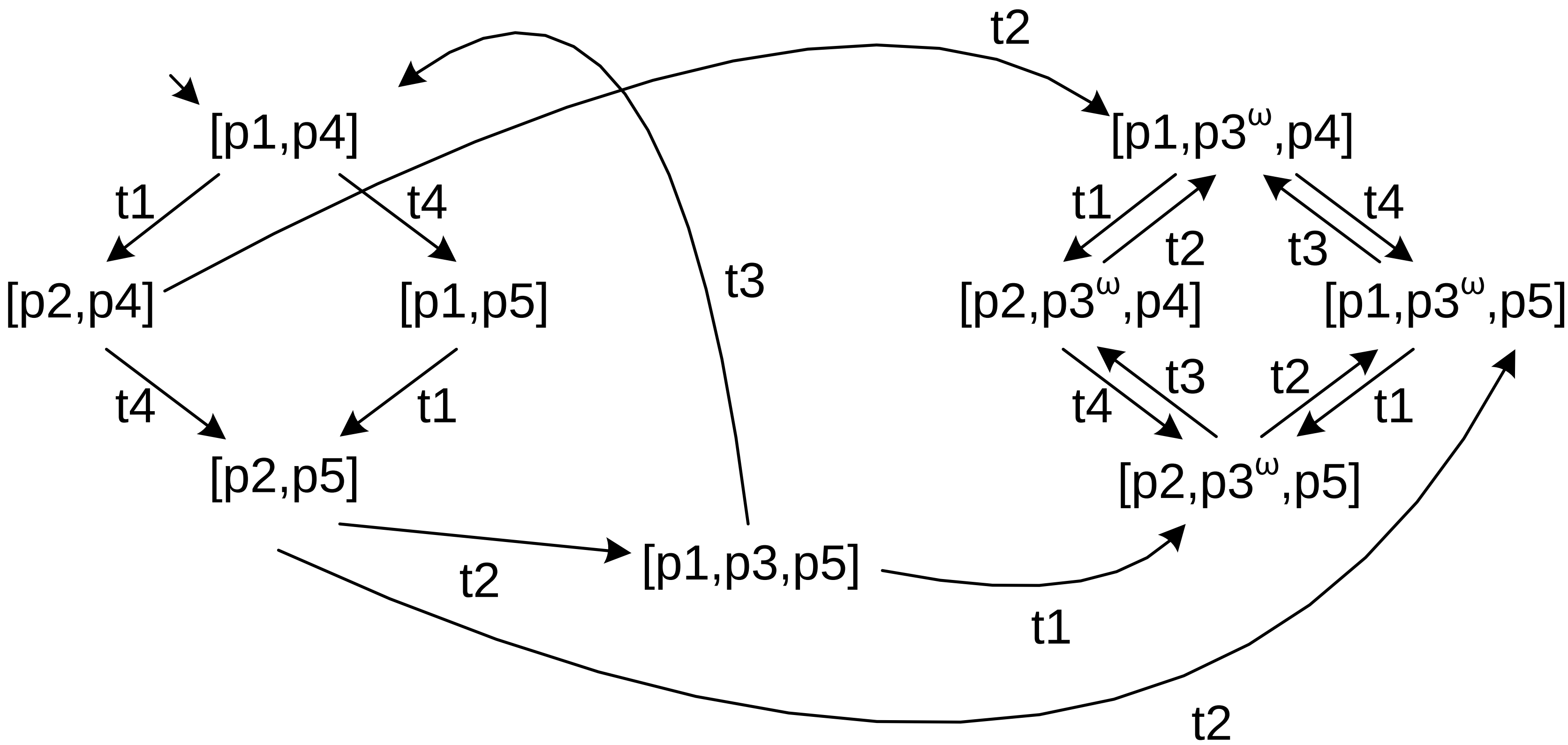


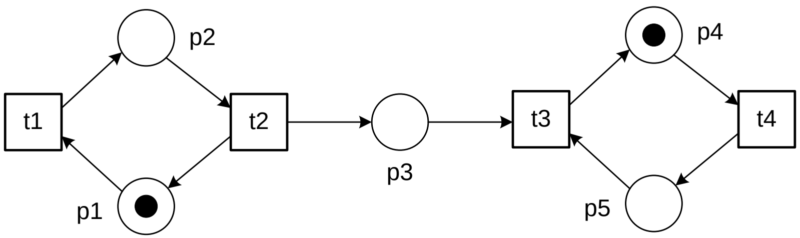
Coverability tree

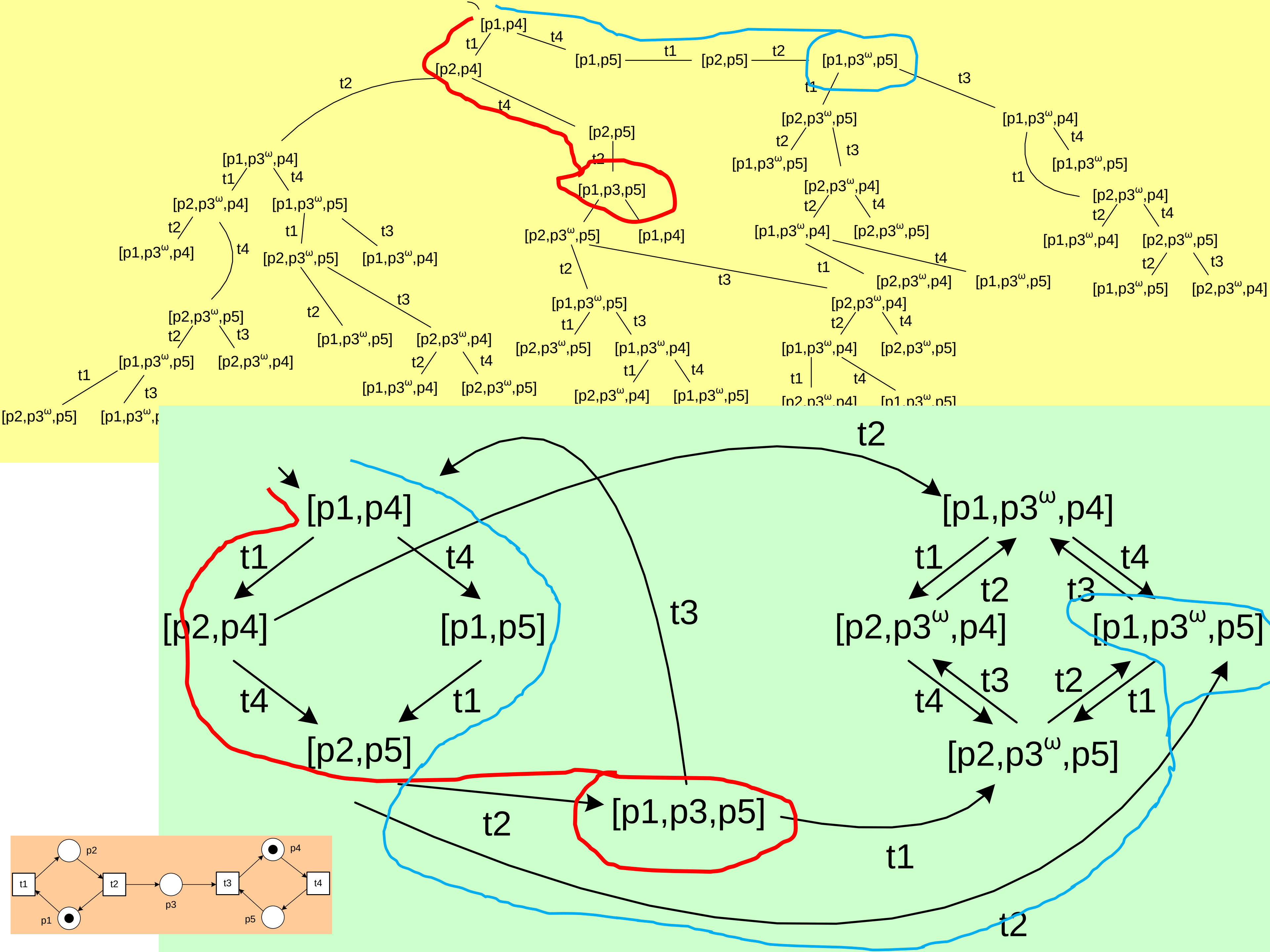




Coverability graph

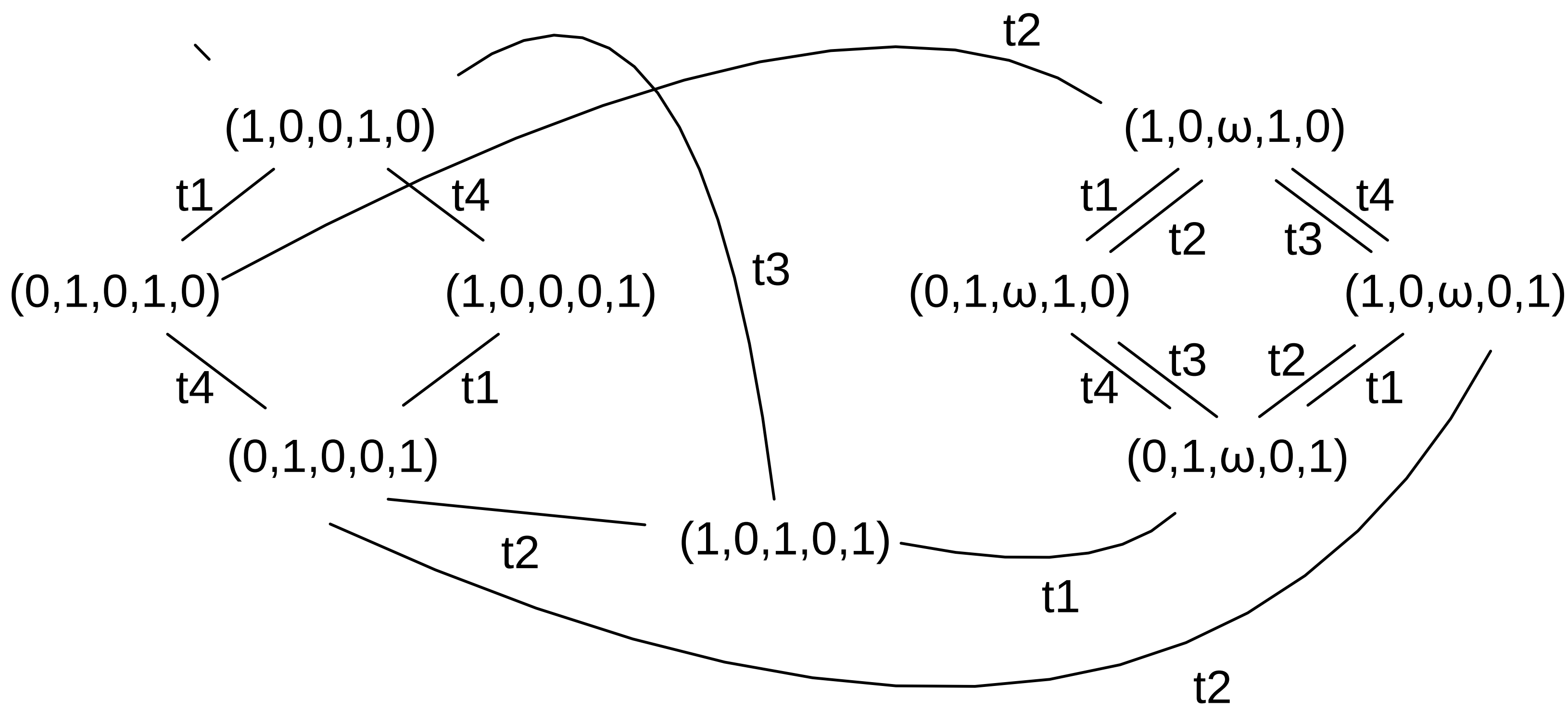


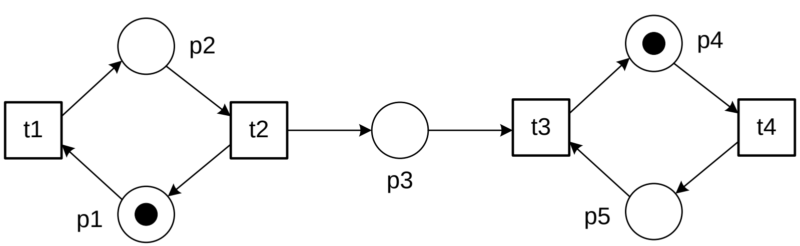




[p1,p4]
t1
t4
[p2,p4]
[p1,p5]
[p2,p5]
[p1,p3ω,p5]
[p1,p3ω,p4]
[p2,p3ω,p4]
[p2,p3ω,p5]
[p1,p3,p5]
[p1,p4]
t2
t3
p1
p2
p3
p4
p5

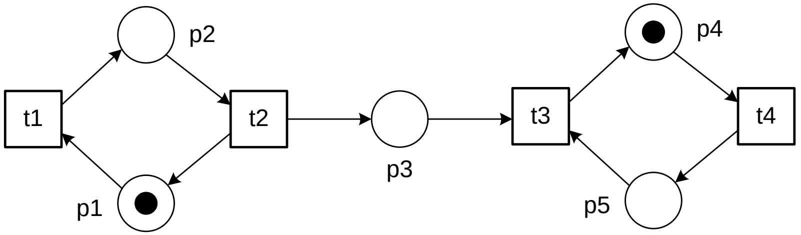
Coverability graph (vector notation)

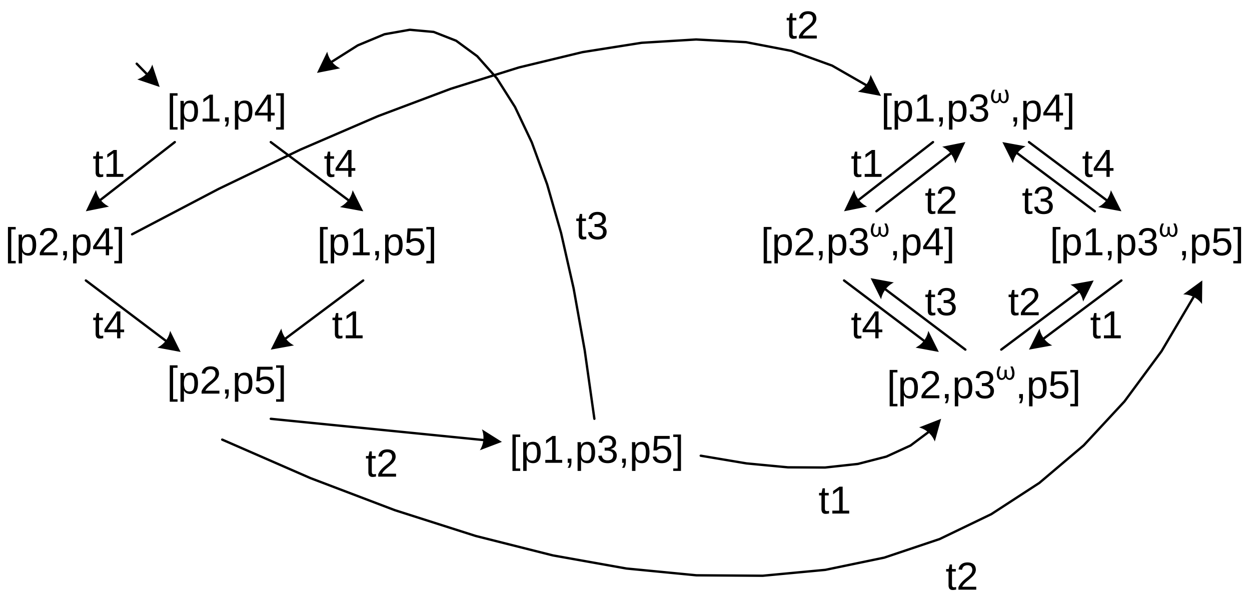




Analysis results

- **p1, p2, p4, p5 are safe**
- **p3 is unbounded**
- **[p2,p5] is reachable**
- **[p1,p2] is not reachable**
- **[p1,p3^{180},p5] is coverable**

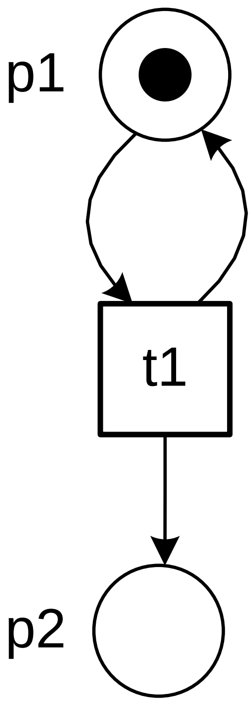




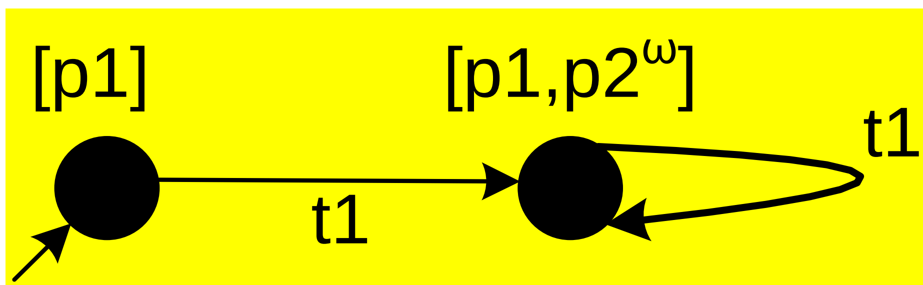
Additional properties

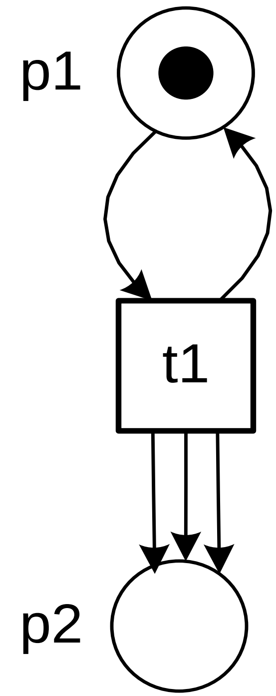
- **A transition *t* is dead if and only if if does not appear in the coverability graph.**
- **The coverability graph and reachability graph are identical if the marked Petri net is bounded (i.e., only ω-free markings).**
- **The marked Petri net is safe if only 0's and 1's appear in nodes.**
- **Any firing sequence of the marked Petri net can be matched by a "walk" through the coverability graph.**
- **The reverse is not true!!!!**

Limitation: Loss of information



Two nets with the same coverability graph!

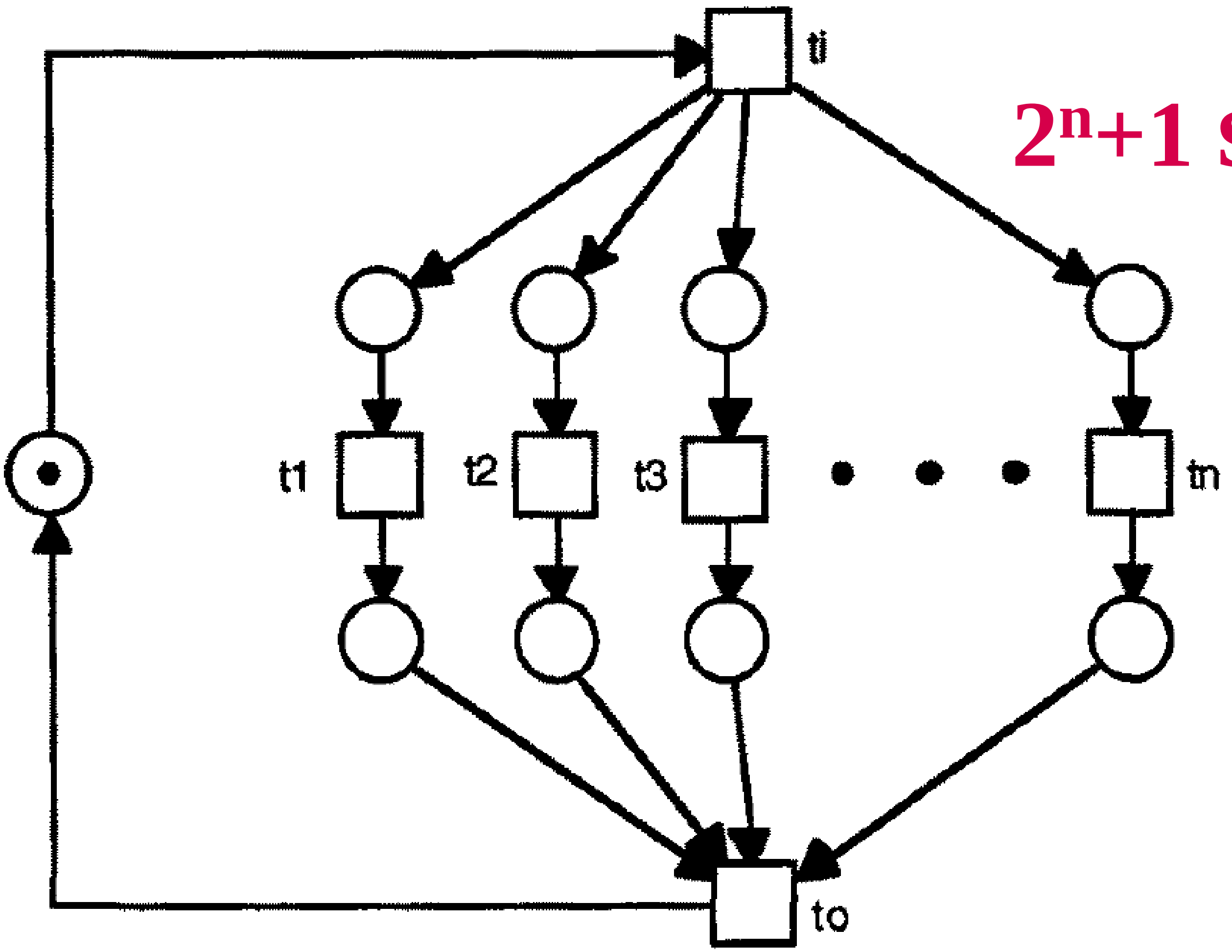




{[p1],[p1,p2^{1}], [p1,p2^{2}], [p1,p2^{3}], [p1,p2^{4}], ...}

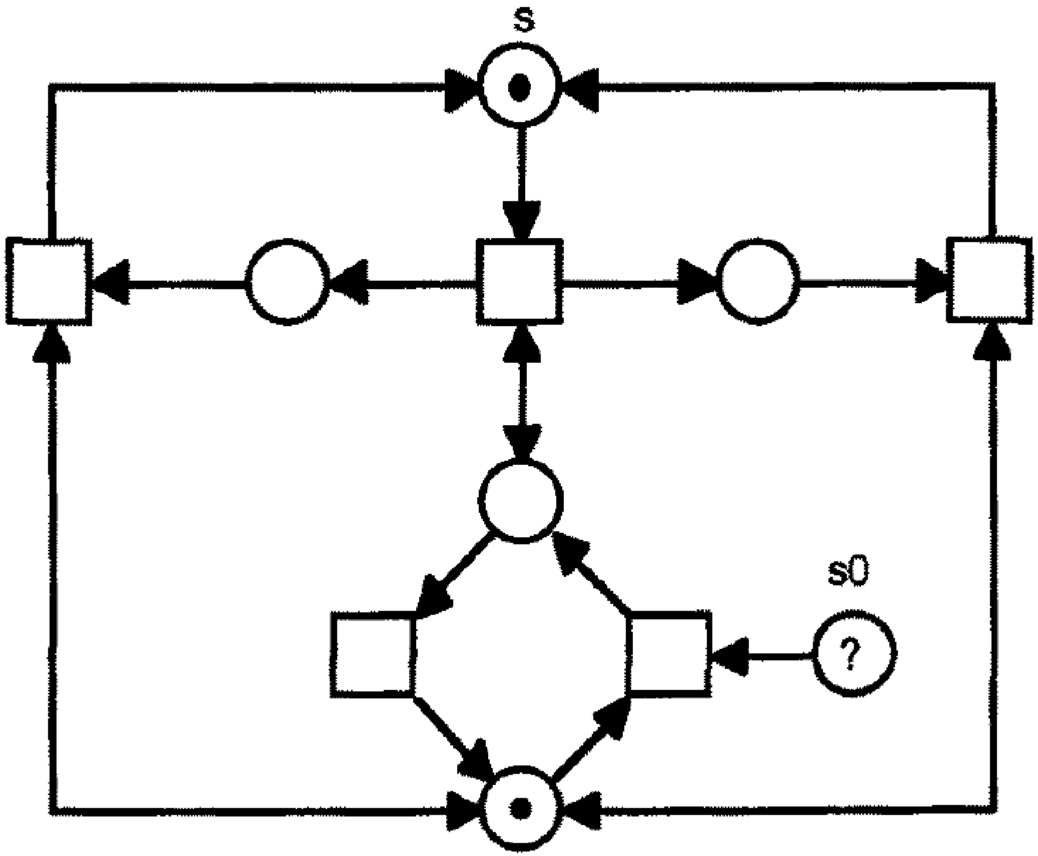
{[p1],[p1,p2^{3}], [p1,p2^{6}], [p1,p2^{9}], [p1,p2^{12}], ...}

State-explosion problem (1)



2^n+1 states

State-explosion problem (2)



place s is
2^n bounded

Each round the number of tokens in s can be doubled.

Variants

- **Construct the coverability graph on the fly (i.e., do not first construct the coverability tree): the graph may become smaller but process is typically non-deterministic.**
- **Several approaches have been proposed to construct "minimal" coverability graphs/sets (see "Alain Finkel: The Minimal Coverability Graph for Petri Nets. Applications and Theory of Petri Nets 1991: 210-243", and "Gilles Geeraerts, Jean-François Raskin, Laurent Van Begin: On the Efficient Computation of the Minimal Coverability Set for Petri Nets. ATVA 2007: 98-113")**

Conclusion

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Where innovation starts

The coverability graph is finite but ...

- some information gets lost in case of unbounded behavior, and
- it may be huge and impossible to construct.

Next: structural methods like invariants, siphons, traps, etc.

After this lecture you should be able to:

- Understand the formalizations, i.e., (P,T,F,W), M, (N,M)[t>(N,M'), etc.
- Determine whether a concrete marked net is terminating, deadlock-free, live, bounded, safe, and/or reversible, whether a transition is live and/or dead, whether a place is k-bounded, etc.
- Construct a Petri net that has a set of desirable properties, e.g., a net that is live and bounded but not reversible.
- Construct the reachability graph of a marked net.
- Construct the coverability tree of a marked net.
- Construct the coverability graph of a marked net.
- Tell which properties can(not) be derived from the coverability tree/graph.
- Understand the limitations of the coverability tree/graph (loss of information, inability to decide liveness, etc.).
- Derive conclusions from a concrete coverability tree/graph.

Appendix: Formalization of Coverability Graph based on Desel & Reisig

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Where innovation starts

Coverability tree & graph

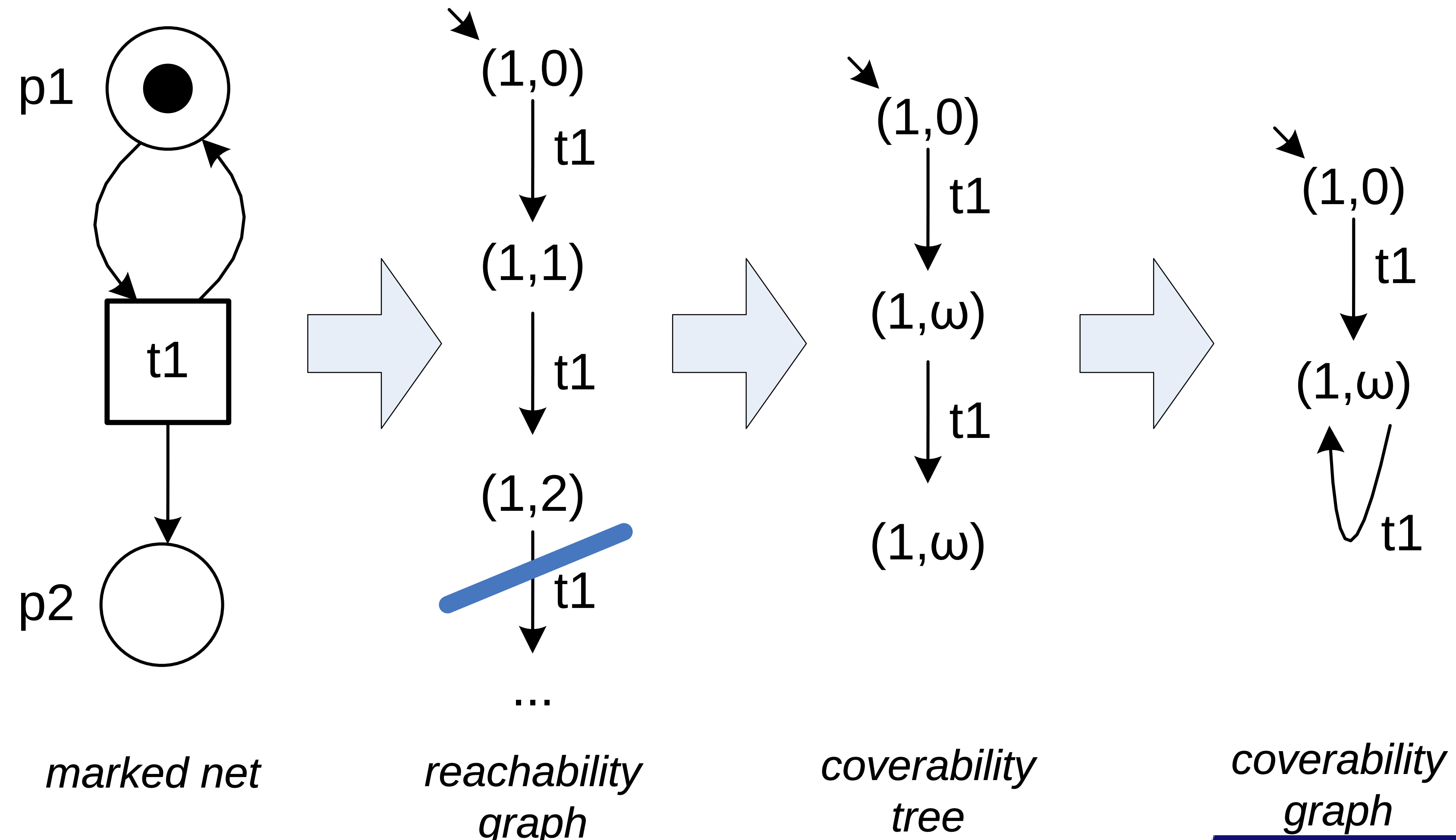
- **Idea: cut-off unbounded behavior using omega (ω) markings**

Formally, an *ω-marking* of a net N is a mapping $\overline{m}: S_N \to \mathbb{N} \cup \{\omega\}$ where $\omega \notin \mathbb{N}$. Clearly, every (conventional) marking can be viewed as a particular ω-marking without ω-entries.

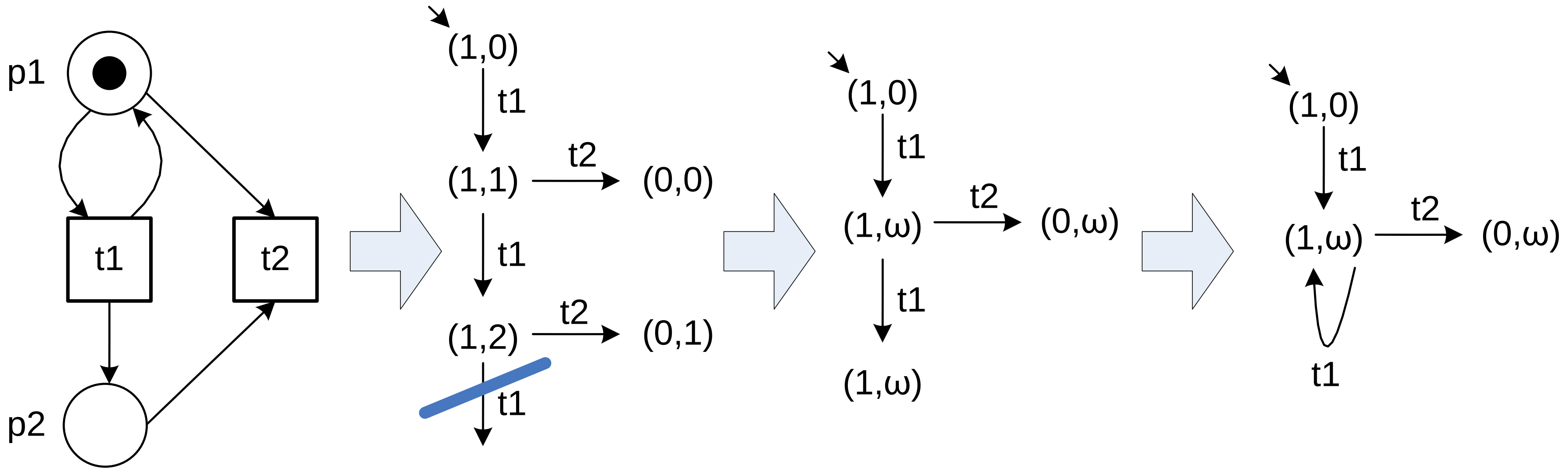
(1,0, ω,1, ω ,1,2,0)

ω-markings are interpreted as follows: If a marking m' is reachable from a marking m and satisfies $m'(s) \geq m(s)$ for each place s, the occurrence sequence leading from m to m' can be iterated arbitrarily often (Proposition 5). If moreover $m'(s_0) > m(s_0)$ for some place s_0 then the number of tokens on s_0 increases with each iteration of the occurrence sequence. This increasing sequence of markings is now replaced by one ω-marking $\overline{m'}$ with $\overline{m'}(s_0) = \omega$, denoting that, for each $b \in \mathbb{N}$, there is a reachable marking that coincides with m' for all places except s_0 and assigns at least b tokens to s_0. More generally, several places may map to ω, representing simultaneous growth of the token count on these places.

Trivial example



Extended example



marked net

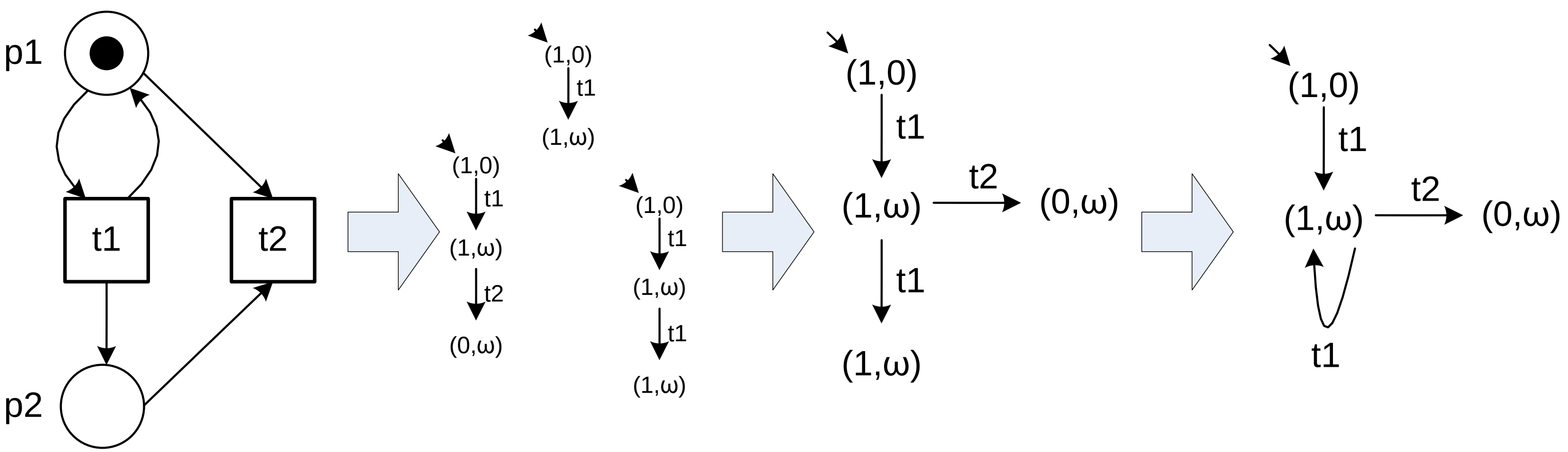
reachability graph

coverability tree

coverability graph

Approach

1. **Define omega (ω) occurrence sequences.**
2. **Show that these are finite.**
3. **Construct coverability tree**
4. **Construct coverability graph**



marked net

ω-occurrence sequences

coverability tree

coverability graph

Example of a ω-occurrence sequence

- **ω-occurrence sequence: t1 t1**
- **(1,0) -t1-> (1,ω) -t1-> (1, ω)**

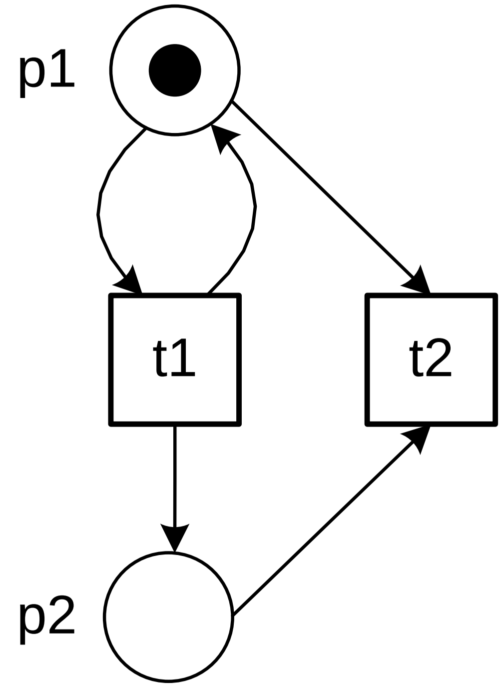
(1,0)

t1

(1,ω)

t1

(1,ω)



marked net

A (finite or infinite) sequence of transitions $t_1\, t_2\, t_3 \ldots$ is an *ω-occurrence sequence* of a marked net with initial marking m_0 if there exist ω-markings $\overline{m}_0, \overline{m}_1, \overline{m}_2, \ldots$ such that m_0 and $\overline{m}_0$ coincide for all places and, for each index i occurring in the sequence $t_1\, t_2\, t_3 \ldots$ the following conditions hold:

(1) For each place s in ${}^\bullet t_i$, either $\overline{m}_{i-1}(s) > 0$ or $\overline{m}_{i-1}(s) = \omega$ (the enabling condition).

(2) For each place s satisfying $\overline{m}_i(s) \neq \omega$,

$$\overline{m}_i(s) = \overline{m}_{i-1}(s) - |F_N \cap \{(s, t_i)\}| + |F_N \cap \{(t_i, s)\}|$$

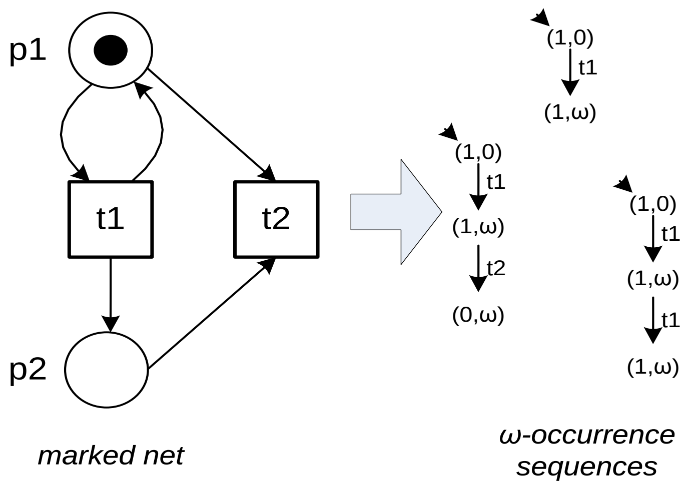
(the conventional marking transformation).

(3) A place s satisfies $\overline{m}_i(s) = \omega$ if and only if
- either $\overline{m}_{i-1}(s) = \omega$
  (places marked by ω remain marked by ω),
- or $\overline{m}_{i-1}(s) \neq \omega$ and there exists an index j, $j < i$, such that $\overline{m}_j(s) \neq \omega$ and $\overline{m}_j(s) < \overline{m}_{i-1}(s) - |F_N \cap \{(s, t_i)\}| + |F_N \cap \{(t_i, s)\}|$ and $\overline{m}_j(s') \leq \overline{m}_{i-1}(s') - |F_N \cap \{(s', t_i)\}| + |F_N \cap \{(t_i, s')\}|$ for each place s' satisfying $\overline{m}_j(s') \neq \omega$ and $\overline{m}_{i-1}(s') \neq \omega$
  (places with increasing token count are marked by ω).

(4) If $i > 1$ then $\overline{m}_{i-1} \notin \{\overline{m}_0, \ldots, \overline{m}_{i-2}\}$
(after reaching an ω-marking the second time, the sequence stops).

We call an ω-marking $\overline{m}$ *reachable* in a marked net if some ω-occurrence sequence leads to $\overline{m}$.

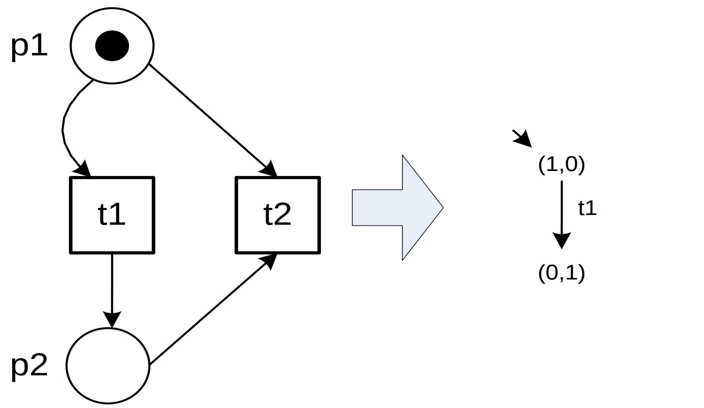
(1) Transitions need to be enabled



Only t1 is enabled in (1,0), not t2.

(1) For each place s in ${}^\bullet t_i$, either $\overline{m}_{i-1}(s) > 0$ or $\overline{m}_{i-1}(s) = \omega$ (the enabling condition).

(2) For non-ω place markings: business as usual



marked net

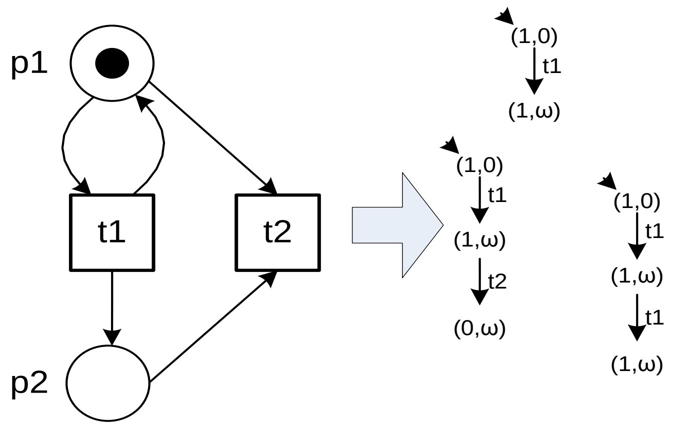
ω-occurrence sequences

(2) For each place s satisfying $\overline{m}_i(s) \neq \omega$,

$$\overline{m}_i(s) = \overline{m}_{i-1}(s) - |F_N \cap \{(s,t_i)\}| + |F_N \cap \{(t_i,s)\}|$$

(the conventional marking transformation).

(3) Introducing omegas



marked net

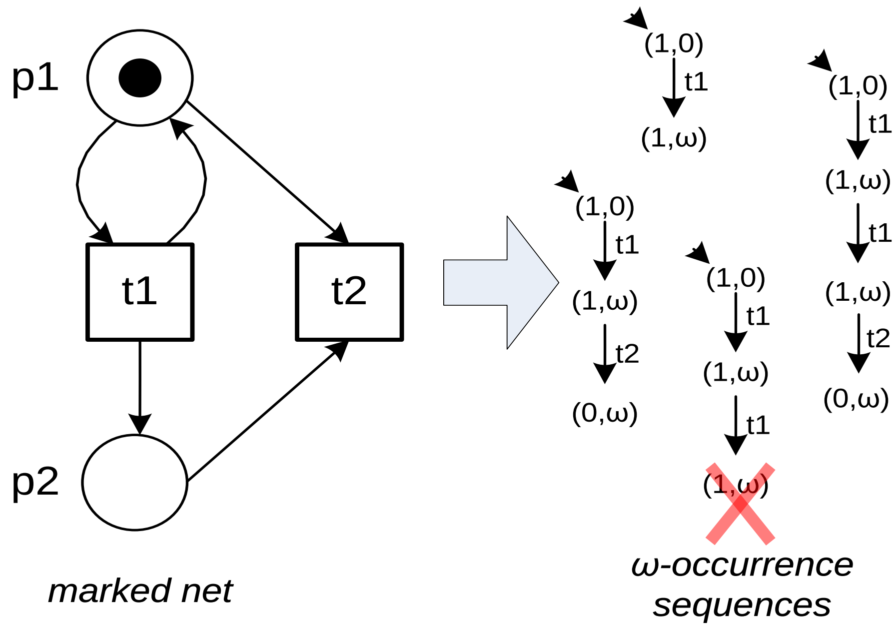
ω-occurrence sequences

(1,ω) is "reachable" from (1,0) because there is a j (j=0) such that ...

(3) A place s satisfies $\overline{m}_i(s) = \omega$ if and only if

- either $\overline{m}_{i-1}(s) = \omega$ (places marked by ω remain marked by ω),
- or $\overline{m}_{i-1}(s) \neq \omega$ and there exists an index j, $j < i$, such that $\overline{m}_j(s) \neq \omega$ and $\overline{m}_j(s) < \overline{m}_{i-1}(s) - |F_N \cap \{(s, t_i)\}| + |F_N \cap \{(t_i, s)\}|$ and $\overline{m}_j(s') \leq \overline{m}_{i-1}(s') - |F_N \cap \{(s', t_i)\}| + |F_N \cap \{(t_i, s')\}|$ for each place s' satisfying $\overline{m}_j(s') \neq \omega$ and $\overline{m}_{i-1}(s') \neq \omega$ (places with increasing token count are marked by ω).

(4) Stop after second identical marking



Marking (0,ω) is dead while (1,ω) markings are not continued after second occurrence.

(4) If $i > 1$ then $\overline{m}_{i-1} \notin \{\overline{m}_0, \ldots, \overline{m}_{i-2}\}$
(after reaching an ω-marking the second time, the sequence stops).

Finite?

- How long can a ω-occurrence sequence be?
- How many ω-occurrence sequences are there?
- Is the coverability tree/graph finite?

Dickson's Lemma (1874-1954)

L E Dickson

Lemma 17. *Let S be a finite set and let $\varphi_1\ \varphi_2\ \varphi_3 \ldots$ be an infinite sequence of mappings from S to $\mathbb{N} \cup \{\omega\}$. There exists an infinite sequence of indices $i_1\ i_2\ i_3\ \ldots$ which is strongly monotonic (i.e., $i_1 < i_2 < i_3 < \cdots$) such that, for each s in S,*

$$\varphi_{i_1}(s) \leq \varphi_{i_2}(s) \leq \varphi_{i_3}(s) \leq \cdots.$$

φ1	φ2	φ3	φ4	φ5	φ6	φ7	φ8	φ9	φ10	φ11	φ12	φ13	φ14
1	1	0	0	0	1	1	0	0	0	1	1	0	0
0	1	1	0	0	0	1	1	0	0	0	1	1	0
0	0	1	1	1	0	0	1	1	1	0	0	1	1
0	0	0	1	1	1	1	1	2	2	2	2	2	3

Proof. We prove the following stronger proposition: For each subset S' of S, there exists an infinite strongly monotonic sequence of indices $i_1, i_2, i_3, \ldots$ such that, for each s in S', $\varphi_{i_1}(s) \leq \varphi_{i_2}(s) \leq \varphi_{i_3}(s) \leq \cdots$. We proceed by induction on the number of elements in S'.

Base. If $S' = \emptyset$ then nothing has to be shown.

Step. Assume $S' \neq \emptyset$ and let $s \in S'$. By the induction hypothesis, there exists an infinite strongly monotonic sequence $i_1, i_2, i_3, \ldots$ such that, for each s' in $S' \setminus \{s\}$,

$$\varphi_{i_1}(s') \leq \varphi_{i_2}(s') \leq \varphi_{i_3}(s') \leq \cdots.$$

Now we restrict the sequence $i_1, i_2, i_3, \ldots$ to indices i_k satisfying

$$\varphi_{i_k}(s) \leq \varphi_{i_{k+1}}(s), \ \ \varphi_{i_k}(s) \leq \varphi_{i_{k+2}}(s), \ \ \varphi_{i_k}(s) \leq \varphi_{i_{k+3}}(s) \ldots$$

Clearly, the obtained sequence $i_{k_1}, i_{k_2}, i_{k_3}, \ldots$ satisfies the required property

$$\varphi_{i_{k_1}}(s) \leq \varphi_{i_{k_2}}(s) \leq \varphi_{i_{k_3}}(s) \leq \cdots$$

for each place s in S'. This sequence is infinite because, for each index i_k, every index i_l in $\{i_{k+1}, i_{k+2}, i_{k+3} \ldots\}$ satisfying

$$\varphi_{i_l}(s) \leq \varphi_{i_{k+1}}(s), \varphi_{i_{k+2}}(s), \varphi_{i_{k+3}}(s) \ldots$$

belongs to the sequence, too. Such an index i_l always exists because every nonempty subset of $I\!N \cup \{\omega\}$ has a minimal element. $\square$

Theorem 18. *Every ω-occurrence sequence of a finite marked net is finite.*

Proof. By contraposition, assume a finite marked net that has an infinite ω-occurrence sequence $t_1\, t_2\, t_3 \ldots$,

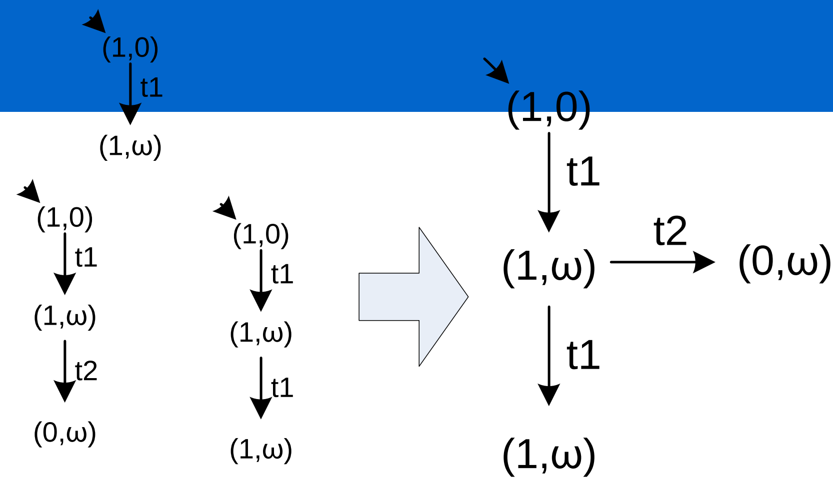
$$\overline{m}_1 \xrightarrow{t_1} \overline{m}_2 \xrightarrow{t_2} \overline{m}_3 \xrightarrow{t_3} \cdots .$$

By Dickson's Lemma (Lemma 17), there exists an infinite strongly monotonic sequence of indices $i_1, i_2, i_3 \ldots$ such that, for each place s,

$$\overline{m}_{i_1}(s) \leq \overline{m}_{i_2}(s) \leq \overline{m}_{i_3}(s) \leq \cdots .$$

Let i and j be two subsequent indices of the sequence $i_1, i_2, i_3 \ldots$. By the definition of ω-occurrence sequences (4) no ω-marking appears twice in an infinite ω-occurrence sequence. Hence $\overline{m}_i(s) \neq \overline{m}_j(s)$ for at least one place s. By the definition of ω-occurrence sequences (3), $\overline{m}_i(s) \neq \omega$ and $\overline{m}_j(s) = \omega$. Again by (3), no place s satisfies $\overline{m}_i(s) = \omega$ and $\overline{m}_j(s) \neq \omega$. Hence $\overline{m}_j$ has more places with ω-entries than $\overline{m}_i$. Therefore, the set of places with ω-entries increases infinitely, contradicting the finiteness of the set of all places of the net. $\square$

Coverability tree



ω-occurrence sequences

coverability tree

Diagrams are a bit misleading: vertices labeled with a ω-marking are really sequences, e.g., initial node is ε rather than (1,0).

Formally, the *coverability tree* of a marked net is defined as a directed graph with a distinguished initial vertex and edges labeled by transitions:

- the vertices are the finite ω-occurrence sequences,
- a distinguished initial vertex is given by the empty sequence ε (which by definition is an ω-occurrence sequence),
- labeled edges are all triples $(\sigma, t, \sigma\, t)$ such that σ as well as $\sigma\, t$ are ω-occurrence sequences.

Finiteness

Theorem 19. *The coverability tree of a finite marked net is finite.*[8]

Proof. By contraposition, assume a finite marked net with an infinite coverability tree. Each vertex σ of the coverability tree has only finitely many immediate successors, one for each transition enabled by the ω-marking reached by σ. Hence every vertex σ with infinitely many successors has at least one immediate successor which also has infinitely many successors. By assumption, the initial vertex ε has infinitely many successors. Hence, starting with ε, we can construct an infinite directed path of the tree. The concatenation of the labels of the edges of this path yields an infinite ω-occurrence sequence — contradicting Theorem 18. □

Corollary 20. *A finite marked net has finitely many reachable ω-markings.* □

Example

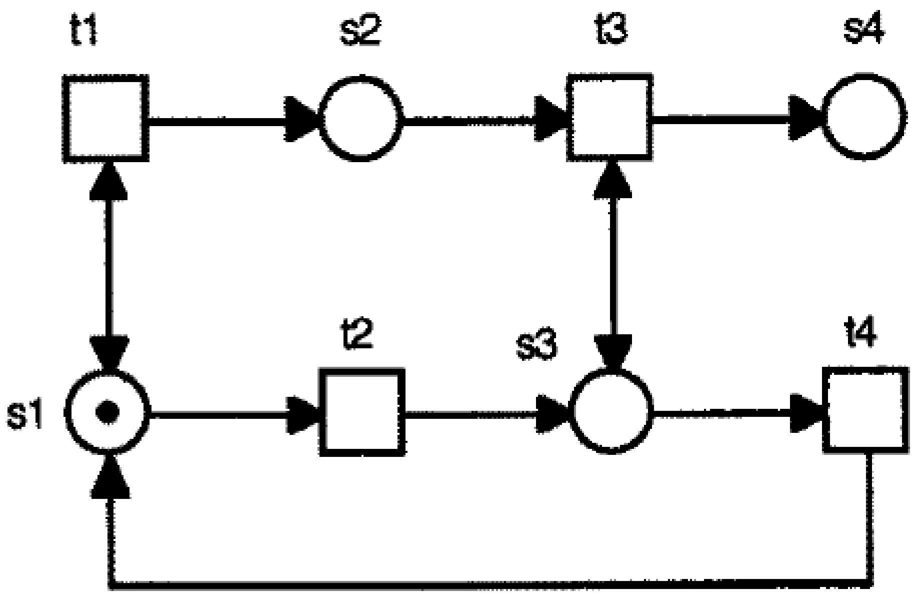
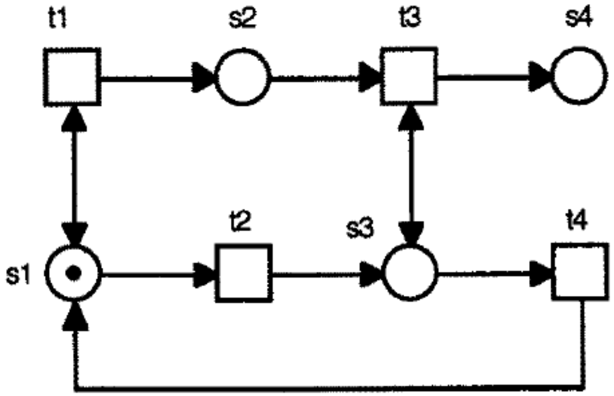
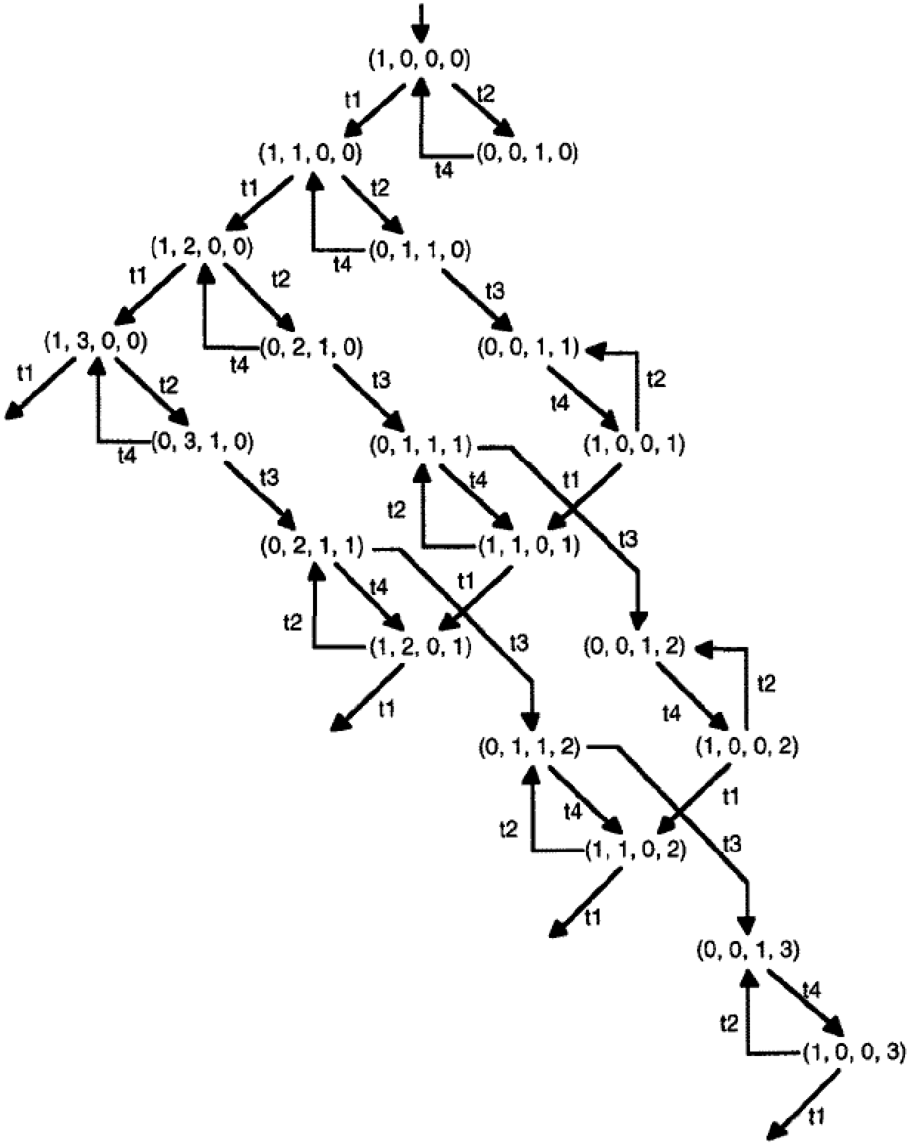


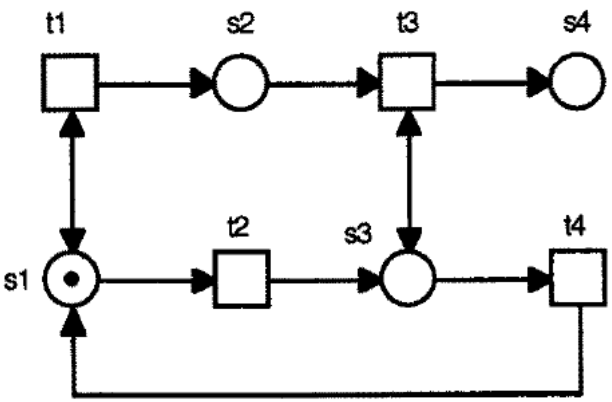
Fig. 12. An unbounded marked Petri net

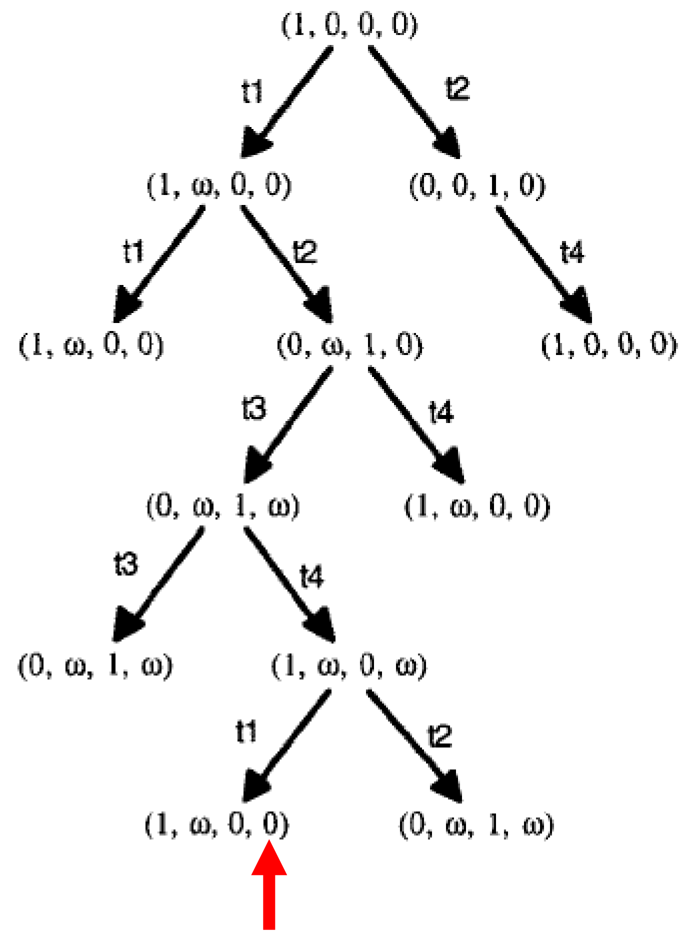
Marking graph
(i.e., reachability graph)





Coverability tree



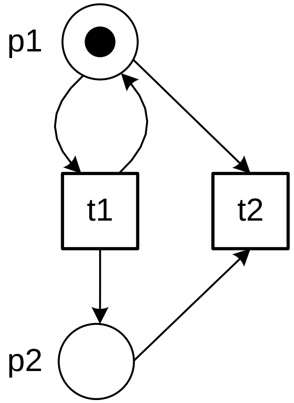


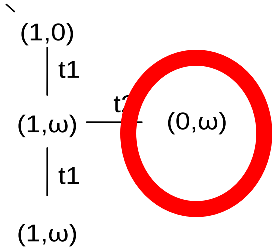
find the error (also in paper)...

Relation ω-markings and normal markings

Theorem 21. *Let $\overline{m}$ be a reachable ω-marking of a finite marked net. For each b in $\mathbb{N}$, there is a reachable marking m such that every place s satisfies:*

- *if $\overline{m}(s) \neq \omega$ then $m(s) = \overline{m}(s)$,*
- *if $\overline{m}(s) = \omega$ then $m(s) \geq b$.*





Let b=180. There is a marking reachable with 0 tokens in p1 and at least 180 tokens in p2.

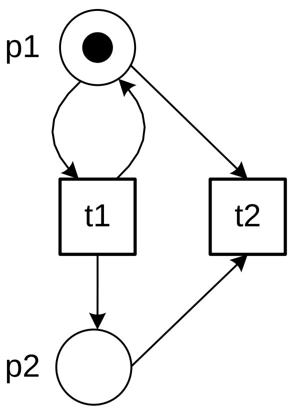
Boundedness = "all ω-markings are ω-free"

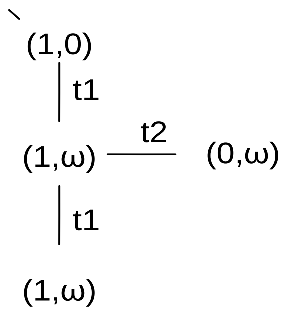
Theorem 23. *A place s of a marked net is not bounded if and only if some reachable ω-marking $\overline{m}$ satisfies $\overline{m}(s) = \omega$ (i.e., some vertex of the coverability tree represents the ω-marking $\overline{m}$).*

Proof.
($\Longleftarrow$) follows immediately from Theorem 21.
($\Longrightarrow$) Since there are only finitely many reachable ω-markings by Theorem 19 there is a number $b \in \mathbb{N}$ such that each reachable ω-marking $\overline{m}$ satisfies either $\overline{m}(s) = \omega$ or $\overline{m}(s) < b$. Since s is not bounded, some reachable marking m satisfies $m(s) \geq b$. Since $m(s)$ does not coincide with $\overline{m}(s)$ for any reachable ω-marking $\overline{m}(s)$, there exists some reachable ω-marking $\overline{m}$ satisfying $\overline{m}(s) = \omega$ by Theorem 22. $\square$

b-boundedness

Corollary 24. *A place s of a marked net is b-bounded if and only if each reachable ω-marking $\overline{m}$ satisfies $\overline{m}(s) \neq \omega$ and $\overline{m}(s) \leq b$.*





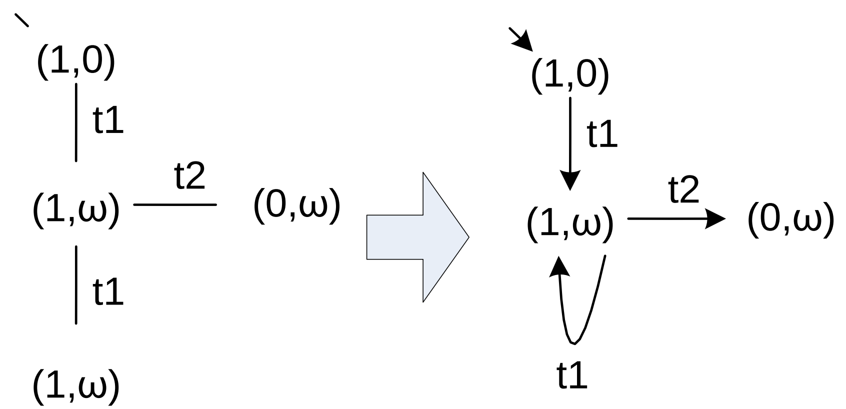
p1 is 1-boundned (safe)
p2 is unbounded

Dead transitions do not appear in cov. tree

Theorem 25. *A transition t of a marked net is dead if and only if t does not occur in any ω-occurrence sequence (i.e., some arc of the coverability tree is labeled by t).*

Proof.
($\Longleftarrow$) Assume some reachable marking m enables t. By Theorem 22, a corresponding reachable ω-marking $\overline{m}$ satisfies $\overline{m}(s) \neq 0$ for each place s in $^\bullet t$. Hence, this ω-marking enables t, too.
($\Longrightarrow$) Assume some reachable ω-marking $\overline{m}$ enables t. By Theorem 21, there is a corresponding reachable marking m that marks all places satisfying $\overline{m} = \omega$ at least once. This marking m enables t, too. $\square$

Coverability graph (versus cov. tree)



coverability tree

coverability graph

The *coverability graph* of a marked net is defined as an arc-labeled directed graph with a distinguished initial vertex and edges labeled by transitions:

- the *vertices* are the reachable ω-markings,
- the distinguished *initial vertex* is given by the ω-marking that coincides with the initial marking for each place,
- labeled edges are given by all triples $(\overline{m}, t, \overline{m}')$ such that $\overline{m}$ and $\overline{m}'$ are reachable ω-markings satisfying $\overline{m} \xrightarrow{t} \overline{m}'$.

Boundedness implies equivalence

Theorem 27. *The coverability graph and the marking graph of a bounded marked net are identical (up to different co-domains of markings and ω-markings).*

Proof. The result follows immediately from Corollary 26 and the definition of ω-occurrence sequences. □

Appendix: Examples taken from Murata

Coverability tree

The coverability tree for a Petri net (N, M_0) is constructed by the following algorithm.

Step 1) Label the initial marking M_0 as the root and tag it "new."

Step 2) While "new" markings exist, do the following:

Step 2.1) Select a new marking M.

Step 2.2) If M is identical to a marking on the path from the root to M, then tag M "old" and go to another new marking.

Step 2.3) If no transitions are enabled at M, tag M "dead-end."

Step 2.4) While there exist enabled transitions at M, do the following for each enabled transition t at M:

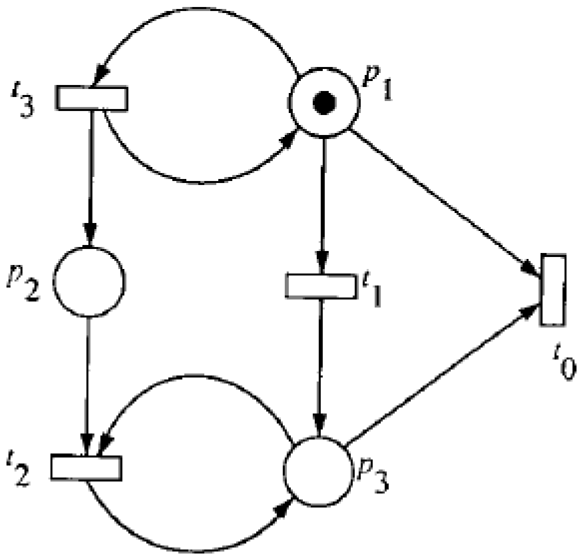
Step 2.4.1) Obtain the marking M' that results from firing t at M.

Step 2.4.2) On the path from the root to M if there exists a marking M'' such that $M'(p) \geq M''(p)$ for each place p and $M' \neq M''$, i.e., M'' is coverable, then replace $M'(p)$ by ω for each p such that $M'(p) > M''(p)$.

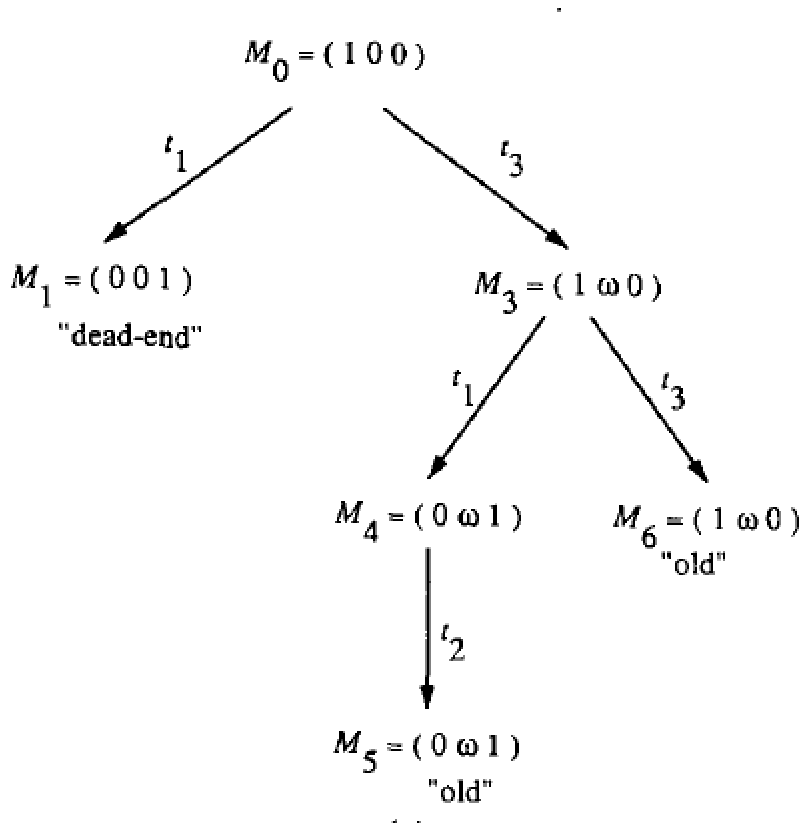
Step 2.4.3) *Introduce* M' as a node, draw an arc with label t from M to M', and tag M' "new."

(same as before)

Example



t_3
p_1
p_2
t_1
t_0
t_2
p_3

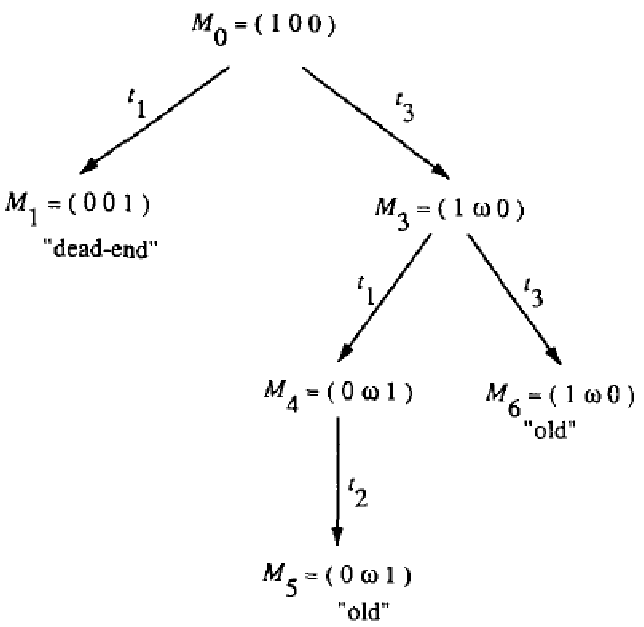


$M_0 = (1\ 0\ 0)$
t_1
t_3
$M_1 = (0\ 0\ 1)$
"dead-end"
$M_3 = (1\ \omega\ 0)$
t_1
t_3
$M_4 = (0\ \omega\ 1)$
$M_6 = (1\ \omega\ 0)$
"old"
t_2
$M_5 = (0\ \omega\ 1)$
"old"

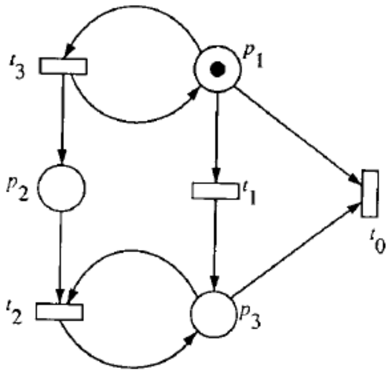
Properties

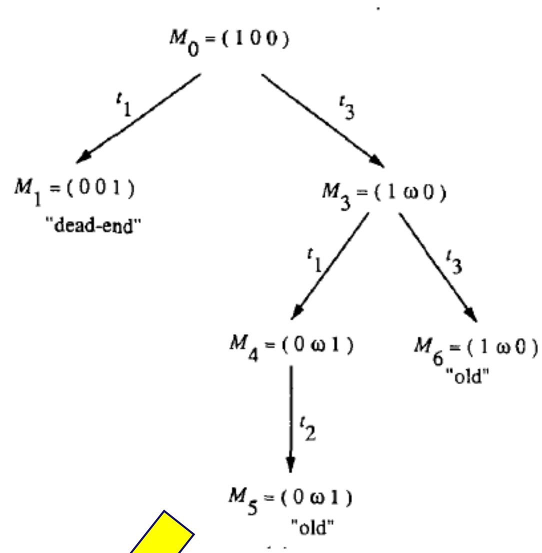
Some of the properties that can be studied by using the coverability tree T for a Petri Net (N, M_0) are the following:

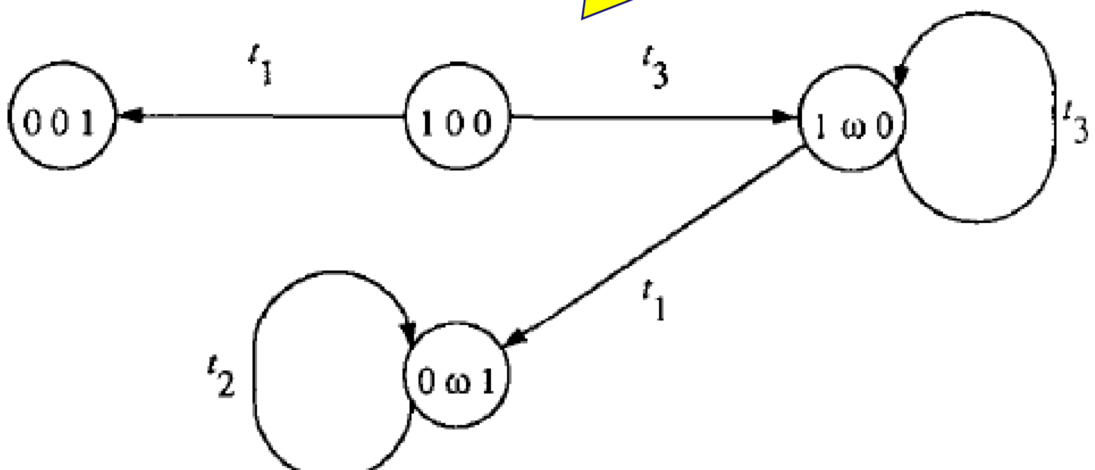
1) A net (N, M_0) is *bounded* and thus $R(M_0)$ is finite *iff* (if and only if) ω does not appear in any node labels in T.
2) A net (N, M_0) is *safe iff* only 0's and 1's appear in node labels in T.
3) A transition t is *dead iff* it does not appear as an arc label in T.
4) If M is reachable from M_0, then there exists a node labeled M' such that $M \leq M'$.



Coverability graph







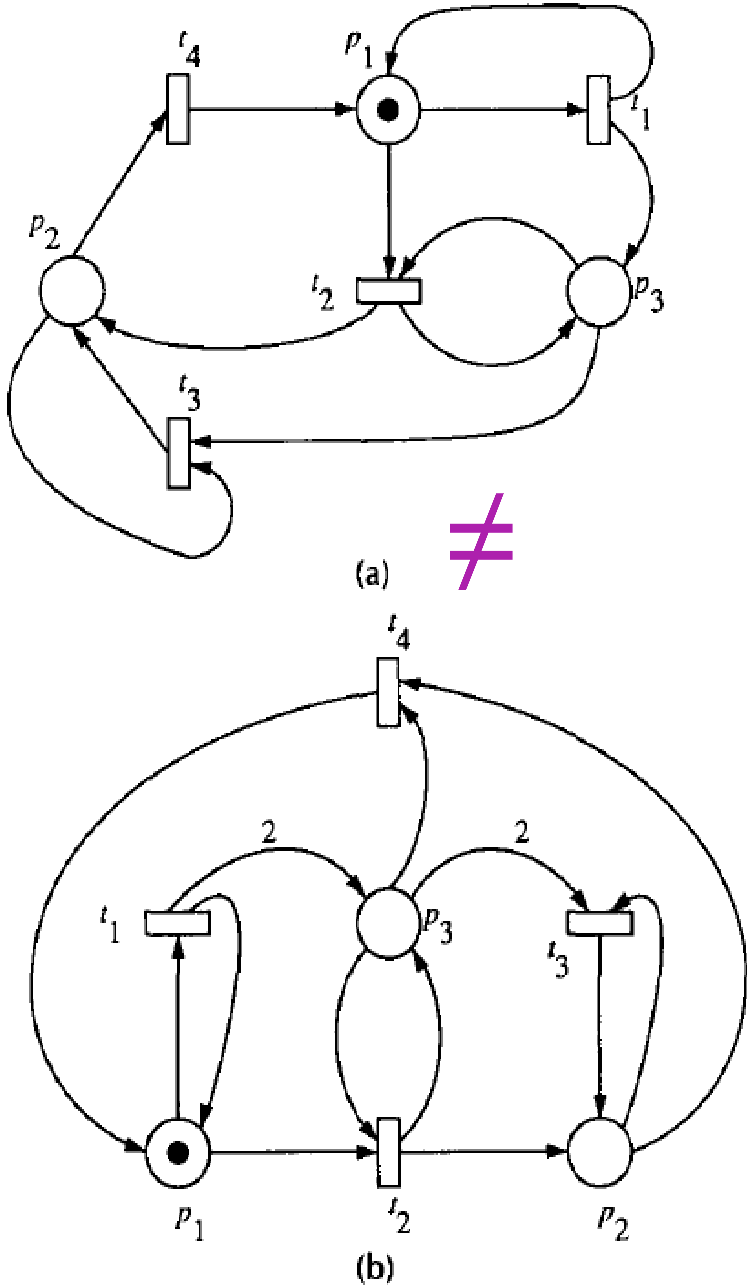


Fig. 19. Two Petri nets having the same converability tre
(a) A live Petri net. (b) A nonlive Petri net.

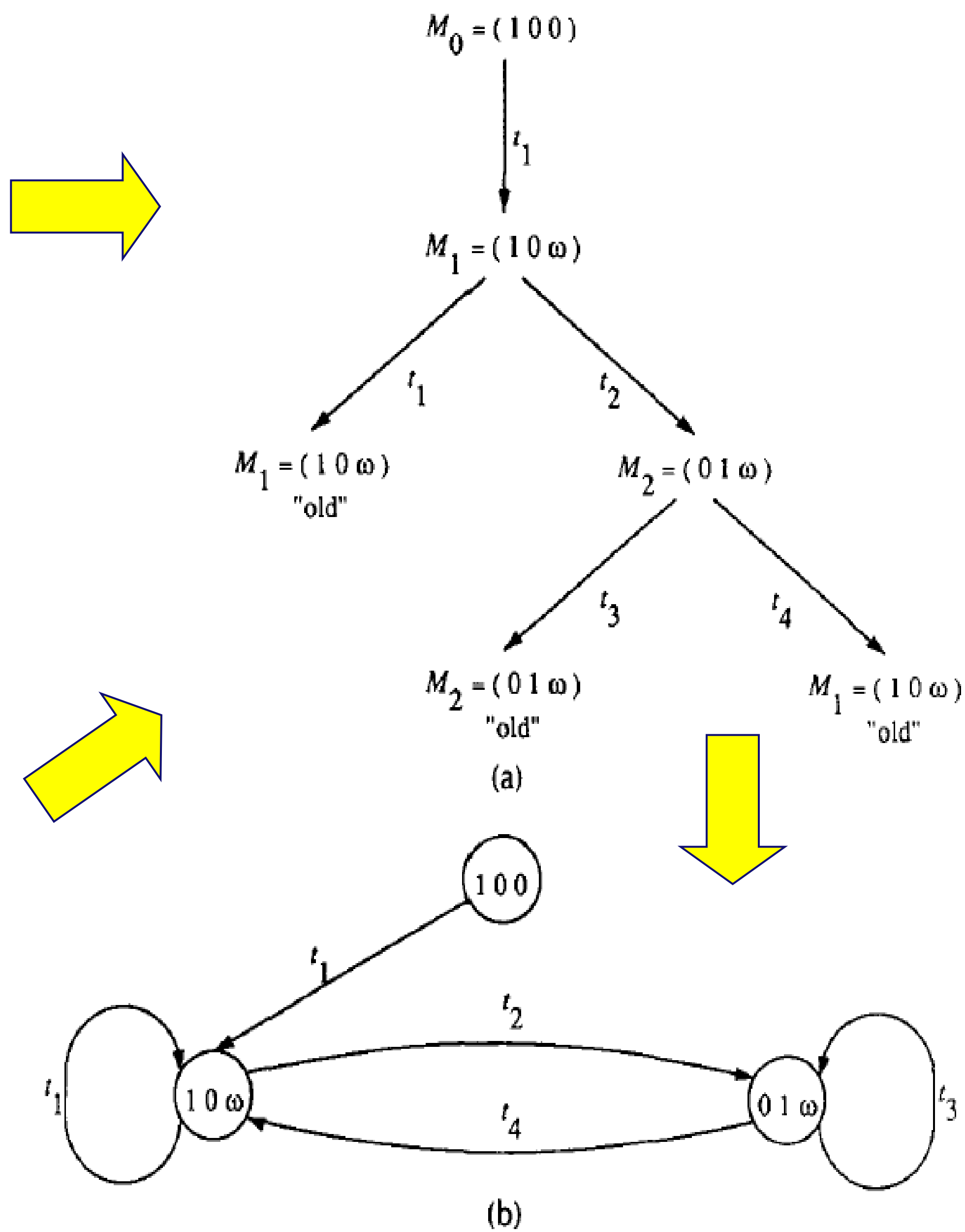


Fig. 20. (a) The coverability tree for both Petri nets shown in Fig. 19(a) and 19(b). (b) The coverability graph for the two nets shown in Fig. 19(a) and 19(b).